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GENERALIZATION OF MULTIPLIERS AND QUASI MULTIPLIERS ON BANACH ALGEBRAS

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Abstract: New notions of (φ, ψ)-multipliers and (φ, ψ)-quasi multipliers on Banach algebras are introduced, where φ and ψ are linear mappings. Examples are given to show that for most of the new notions, the corresponding class of these operators is larger than that for the old versions. General theory is developed for these notions.

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1. Introduction

Ronald Larsen introduced the subject of multipliers on commutative Banach algebras [4]. After him various versions of multipliers on Banach algebras defined, such as quasi-multiplies and φ-multipliers.

Let A be a Banach algebra and $\varphi : A \longrightarrow A$ a bounded linear map. The concept of φ-multipliers defined by Riazi and Adib in [8]. A left (resp. right) φ-multiplier on A is a bounded linear translation $T : A \longrightarrow A$ such that $T(xy) = T(x)\varphi(y)$ (resp. $T(xy) = \varphi(x)T(y)$) for all $x, y \in A$. The collection of all left and right φ-multipliers denoted by $M_{\varphi l}(A)$ and $M_{\varphi r}(A)$, respectively. Linear mapping T called a φ-multiplier on A if it is both a left φ-multiplier and a right φ-multiplier on A, and the collection of all φ-multiplier on A denoted by $M_\varphi(A)$.

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A quasi-multiplier is a generalization of the notion of a left (right, double) multiplier and was first introduced by Akemann and Pedersen [2]. McKennon in [7], studied systematic account of the general theory of quasi-multipliers on a Banach algebra with a bounded approximate identity.

Let A be a Banach algebra. A bilinear map $m : A \times A \longrightarrow A$ is quasi multiplier if satisfies

$$m(ab, cd) = am(b, c)d, \tag{1.1}$$

for every $a, b, c \in A$. The notion of quasi multipliers considered by many authors in many literatures such as [1, 3, 5, 6].

A Banach algebra A is called without order if for all $a \in A$, $aA = \{0\}$ implies $a = 0$, or, for all $a \in A$, $Aa = \{0\}$ implies $a = 0$.

In this paper, at first we consider (φ, ψ)-multipliers on Banach algebras, and in the third section, by introducing (φ, ψ)-quasi multipliers, we studied properties of this new notion.

2. (φ, ψ)-MULTIPLIERS

Definition 2.1. *Let A be a Banach algebra and let $\varphi, \psi : A \longrightarrow A$ be linear maps. We say a linear map $T : A \longrightarrow A$ is (φ, ψ)-multiplier if satisfies*

$$T(abc) = \varphi(a)T(b)\psi(c), \tag{2.1}$$

for every $a, b, c \in A$. The collection of (φ, ψ)-multipliers denoted by $M_{\varphi,\psi}(A)$. In other word, a (φ, ψ)-multiplier, is a left ψ-multiplier and right φ-multiplier on Banach algebra A. We denote the collection of right φ-multipliers on A by $M_{\varphi r}(A)$ and the collection of left ψ-multipliers on A by $M_{\psi l}(A)$.

Example 2.2.

1. *Let A be a Banach algebra and T be a left and right multiplier on A. Then T is (id_A, id_A)-multiplier, where id_A is identity map on the algebra A.*
2. *Let A be a Banach algebra and $\varphi : A \longrightarrow A$ be a homomorphism. Then φ is a (φ, φ)-multiplier.*
3. *Let A be a factorable Banach algebra and $\varphi : A \longrightarrow A$ be 3-homomorphism. Then φ is a (φ, φ)-multiplier.*

Theorem 2.3. *Let A be a commutative Banach algebra and φ and ψ be bounded idempotent homomorphisms. Then $M_{\varphi,\psi}(A)$ is a Banach algebra respect to operator norm. Moreover if $A^2 = A$, $\varphi \circ T = T \circ \varphi$ and $\psi \circ T = T \circ \psi$ for all $T \in M_{\varphi,\psi}(A)$, then $M_{\varphi,\psi}(A)$ is commutative and without order.*

Proof. According to the definition 2.1, $M_{\varphi,\psi}(A) \subseteq B(A)$ (where $B(A)$ will denote the Banach algebra of all continuous linear operators from A into A) and it is easy show that $M_{\varphi,\psi}(A)$ is a linear space. We show that $M_{\varphi,\psi}(A)$ is a subalgebra of $B(A)$ and then $M_{\varphi,\psi}(A)$ is an algebra.

Let $T_1, T_2 \in M_{\varphi,\psi}(A)$ then

$$\begin{aligned}(T_1 \circ T_2)(xyz) &= T_1(T_2(xyz)) = T_1(\varphi(z)T_2(y)\psi(x)) \\ &= \varphi(\varphi(x))T_1(T_2(y))\psi(\psi(z)) \\ &= \varphi^2(x)(T_1 \circ T_2)(y)\psi^2(z) = \varphi(x)(T_1 \circ T_2)(y)\psi(z).\end{aligned}$$

Hence $T_1 \circ T_2 \in M_{\varphi,\psi}(A)$ and $M_{\varphi,\psi}(A)$ is an algebra. Clearly, $M_{\varphi,\psi}(A)$ is closed in $B(A)$, and thereby $M_{\varphi,\psi}(A)$ is Banach algebra.

By Example 2.2, $\varphi, \psi \in M_{\varphi,\psi}(A)$, therefore $M_{\varphi,\psi}(A) \neq \{0\}$. Fix $S, T \in M_{\varphi,\psi}(A)$. By commutativity of A we have

$$\begin{aligned}(T \circ S)(xyz) &= T(S(xyz)) = T(\varphi(x)S(y)\psi(z)) = T(S(y)\psi(z)\varphi(x)) \\ &= \varphi(S(y))T(\psi(z))\psi(\varphi(x)) = S(\varphi(y))\psi(T(z))\varphi(\psi(x)) \\ &= \varphi(\psi(x))S(\varphi(y))\psi(T(z)) = S(\psi(x)\varphi(y)T(z)) \\ &= S(\varphi(y)T(z)\psi(x)) = S(T(yzx)) = (S \circ T)(yzx) \\ &= (S \circ T)(xyz),\end{aligned}$$

for all $x, y, z \in A$. Thus $M_{\varphi,\psi}(A)$ is commutative.

Now we show that $M_{\varphi,\psi}(A)$ is without order. Let $T \in M_{\varphi,\psi}(A)$ such that $T \circ S = 0$, for each $S \in M_{\varphi,\psi}(A)$. By using the fact that $\psi \in M_{\varphi,\psi}(A)$, hence $T \circ \psi = 0$ and

$$\begin{aligned}T(xyz) &= \varphi(x)T(y)\psi(z) = \varphi(\varphi(x))T(y)\psi(\psi(z)) \\ &= T(\varphi(x)y\psi(z)) = T(\varphi(x)\psi(z)y) \\ &= \varphi(\varphi(x))T(\psi(z))\psi(y) = 0,\end{aligned}$$

for all $x, y, z \in A$. This follows $T(xyz) = 0$, for all $x, y, z \in A$. On the other hand, since $A^2 = A$, then $T(x) = T(xx) = T(xxx) = 0$. This means that $T(x) = 0$, for all $x \in A$, and so $T = 0$. Thus $M_{\varphi,\psi}(A)$ is without order. □

Theorem 2.4. *let A be a Banach algebra and φ, ψ be isomorphisms from A into A. If $T \in M_{\varphi r}(A) \cap M_{\psi l}(A)$ then the following statements are equivalent.*

1. *T is bijective.*

2. *T^{-1} exists and $T^{-1} \in M_{\varphi^{-1},\psi^{-1}}(A)$.*

Proof. Obviously (2) implies (1). If (1) holds, then by inverse mapping theorem T^{-1} exists and $T^{-1} \in B(A)$. Moreover if $x, y, z \in A$ then

$$\begin{aligned} T^{-1}(x)\varphi^{-1}(y) &= T^{-1} \circ T(T^{-1}(x)\varphi^{-1}(y)) \\ &= T^{-1}(T(T^{-1}(x)\varphi^{-1}(y))) \\ &= T^{-1}(T(T^{-1}(x))\varphi(\varphi^{-1}(y))) \\ &= T^{-1}(xy). \end{aligned}$$

A similar computation shows that

$$\varphi^{-1}(x)T^{-1}(y) = T^{-1}(xy), \quad T^{-1}(x)\psi^{-1}(y) = T^{-1}(xy).$$

Therefore

$$\begin{aligned} \varphi^{-1}(x)T^{-1}(y)\psi^{-1}(z) &= T^{-1}(xy)\psi^{-1}(z) \\ &= T^{-1}((xy)z) = T^{-1}(xyz). \end{aligned}$$

Thus $T^{-1} \in M_{\varphi^{-1},\psi^{-1}}(A)$. □

3. (φ, ψ)-QUASI MULTIPLIERS

In this section, we establish some basic properties of (φ, ψ)-quasi multipliers on Banach algebras. We start by the following definition.

Definition 3.1. *Let A be a Banach algebra and let $\varphi, \psi : A \longrightarrow A$ be linear maps. We say a map $m : A \times A \longrightarrow A$ is (φ, ψ)-quasi multiplier if satisfies*

$$m(ab, cd) = \varphi(a)m(b, c)\psi(d) \tag{3.1}$$

for every $a, b, c, d \in A$.

A wide range of examples are as follows:

Example 3.2.

1. *Every quasi multiplier is $(id_{\mathcal{A}}, id_{\mathcal{A}})$-quasi multiplier, where $id_{\mathcal{A}}$ is the identity map on the algebra $\mathcal{A}$.*
2. *Let A be an algebra and φ be a homomorphism from A into itself. Define $m : A \times A \longrightarrow A$ by $m(a, b) = \varphi(ab)$. Then m is (φ, φ)-quasi multiplier.*
3. *Let A be an algebra, let φ and ψ be homomorphisms from A into itself. Define $m : A \times A \longrightarrow A$ by $m(a, b) = \varphi(a)\psi(b)$. Then m is (φ, ψ)-quasi multiplier.*

4. *Let φ be an endomorphism on algebra A and x lies in the center of A. Then $m : A \times A \longrightarrow A$ defined by $m(a,b) = x\varphi(ab)$ is (φ,φ)-quasi multiplier. It is clear that if the algebra A is commutative, then every element of A is central and the above defined mapping m is (φ,φ)-quasi multiplier, for every $a,b,x \in A$.*

Above Example shows that the space of new defined version is larger than old version of quasimultipliers.

Theorem 3.3. *Let A be a factorable Banach algebra, and let $\varphi,\psi : A \longrightarrow A$ be homomorphisms and $m : A \times A \longrightarrow A$ be (φ,ψ)-quasi multiplier. Then*

(i) $m(ab,c) = \varphi(a)m(b,c)$ and $m(a,bc) = m(a,b)\psi(c)$, for every $a,b,c \in A$.

(ii) m is bilinear.

(iii) If φ and ψ are continuous, then m is separately and jointly continuous.

Proof. Let $a,b,c \in A$. Since A is factorable, so there are $w,x,y,z \in A$ such that $a = wx$, $y = wy$, and $z = wz$.

For (i) we have:

$$\begin{aligned} m(ab,c) &= m(a(wy),wz) = m((aw)y,wz) = \varphi(aw)m(y,w)\psi(z) \\ &= \varphi(a)\varphi(w)m(y,w)\psi(z) = \varphi(a)m(wy,wz) \\ &= \varphi(a)m(b,c), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} m(a,bc) &= m(wx,(wy)c) = m(wx,w(yc)) = \varphi(w)m(x,w)\psi(yc) \\ &= \varphi(w)m(x,w)\psi(y)\psi(c) = m(wx,wy)\psi(c) \\ &= m(a,b)\psi(c). \end{aligned} \tag{3.3}$$

(ii) For every $\alpha \in \mathbb{C}$,

$$\begin{aligned} m(\alpha a,b+c) &= m(\alpha wx,wy+wz) = m(\alpha wx,w(y+z)) \\ &= \varphi(\alpha w)m(x,w)\psi(y+z) \\ &= \alpha m(a,b) + \alpha m(a,c). \end{aligned} \tag{3.4}$$

Again by factorability of A, there are $p,q,r,v \in A$ such that $a = pv$, $b = qv$, and $c = rv$

$$m(a+b,\alpha c) = m(pv+qv,\alpha pr) = m((p+q)v,p(\alpha r))$$

$$\begin{aligned} &= \varphi(p+q)m(v,p)\psi(\alpha r) \\ &= \alpha m(a,c) + \alpha m(b,c). \end{aligned} \tag{3.5}$$

Hence, the relations (3.4) and (3.5) imply that m is bilinear.

(iii) At first, we show that m is separately continuous. Let $x_0, y_0 \in A$ we show the functions $x \mapsto m(x, y_0)$ and $y \mapsto m(x_0, y)$ are continuous. If $\{x_n\}$ and $\{y_n\}$ are sequences in A such that $x_n \longrightarrow x$, and $y_n \longrightarrow y$. We have to show that $m(x_0, y_n) \longrightarrow m(x_0, y)$, and $m(x_n, y_0) \longrightarrow m(x, y_0)$.

Since $\{x_n - x\}$ and $\{y_n - y\}$ converge to 0 and factorability of A yields there are sequences $\{t_n\}$, $\{w_n\}$, and elements $t, w \in A$ such that

$$\lim_n \|w_n\| = \lim_n \|t_n\| = 0,$$

and

$$x_n - x = t_n t, \qquad y_n - y = w w_n.$$

Then

$$\begin{aligned} \lim_n \|m(x_n, y_0) - m(x, y_0)\| &= \lim_n \|m(x_n - x, y_0)\| = \lim_n \|m(t_n t, y_0)\| \\ &= \lim_n \|\varphi(t_n) m(t, y_0)\| \\ &\leq \lim_n \|\varphi(t_n)\| \|m(t, y_0)\| = 0. \end{aligned} \tag{3.6}$$

Analogously, with continuity of ψ, one can show that

$$\lim_n \|m(x_0, y_n) - m(x_0, y)\| = 0. \tag{3.7}$$

Hence, by (3.6) and (3.7), m is separately continuous. Finally, we show that m is jointly continuous. By above defined notations, we have

$$\begin{aligned} \lim_n \|m(x_n, y_n) - m(x, y)\| &\leq \lim_n \|m(x_n - x, y_n - y)\| \\ &+ \lim_n \|m(x, y_n) - m(x, y)\| \\ &+ \lim_n \|m(x_n, y) - m(x, y)\| \\ &= \lim_n \|m(t_n t, w w_n)\| = \lim_n \|\varphi(t_n) m(t, w) \psi(w_n)\| \\ &\leq \lim_n \|\varphi(t_n)\| \|m(t, w)\| \|\psi(w_n)\| = 0. \end{aligned} \tag{3.8}$$

Hence, m is jointly continuous. □

Corollary 3.4. *Let A be a Banach algebra with bounded approximate identity, and let $\varphi, \psi : A \longrightarrow A$ be homomorphisms and $m : A \times A \longrightarrow A$ be (φ, ψ)-quasi multiplier. Then* *(i)* $m(ab,c)=\varphi(a)m(b,c)$ *and* $m(a,bc)=m(a,b)\psi(c)$, *for every* $a,b,c\in A$.

(ii) m *is bilinear.*

(iii) If φ *and* ψ *are continuous, then* m *is separately and jointly continuous.*

Corollary 3.5. *Let* A *be a factorable Banach algebra, and let* $\varphi : A \longrightarrow A$ *be a homomorphism and* $T \in M_{\varphi r}(A)$ *and* $T' \in M_{l\varphi}(A)$*. If* φ *is continuous, then* T *and* T' *are continuous.*

Let A be a Banach algebra and $\varphi,\psi : A \longrightarrow A$ be linear maps, the set of all bilinear (φ,ψ)-quasimultipliers on A will be denoted by $\mathbf{Q}M_{(\varphi,\psi)}(A)$. Under the usual pointwise operations, $\mathbf{Q}M_{(\varphi,\psi)}(A)$ is a vector (linear) space.

Theorem 3.6. *Let* A *be a Banach algebra, and* $\varphi,\psi : A \longrightarrow A$ *be none zero continuous linear maps. Consider* $\boldsymbol{Q}M_{(\varphi,\psi)}(A)$ *with the following norm*

$$|||m||| = \|\varphi\|\|\psi\| \sup\big\{\|m(a,b)\| : a,b\in A, \|a\|=\|b\|=1\big\},$$

for every $m \in \boldsymbol{Q}M_{(\varphi,\psi)}(A)$*. Then* $(\boldsymbol{Q}M_{(\varphi,\psi)}(A), |||\cdot|||)$ *is a Banach space.*

Proof. At first, we prove that $|||\cdot|||$ is a norm on $\mathbf{Q}M_{(\varphi,\psi)}(A)$. Let $m \in \mathbf{Q}M_{(\varphi,\psi)}(A)$, then clearly $|||m|||\geq 0$, and if $|||m|||=0$ then for all $a,b\in A$ such that $||a||=||b||=1$, we have $\|m(a,b)\|=0$. This implies that $m(a,b)=0$, for all $a,b\in A$ such that $||a||=||b||=1$. Let $(x,y)\in A\times A$ be arbitrary. If x or y is zero then $m(x,y)=0$. Thus suppose that $x,y\neq 0$, then

$$\|m(x,y)\| = \|m(\|x\|\frac{x}{\|x\|}, \|y\|\frac{y}{\|y\|})\| = \|x\|\|y\|\|m(\frac{x}{\|x\|}, \frac{y}{\|y\|})\| = 0, \tag{3.9}$$

this means that $m=0$. Let $\alpha\in\mathbb{C}$, then

$$\begin{aligned} |||\alpha m||| &= \|\varphi\|\|\psi\|\sup\{\|\alpha m(a,b)\| : a,b\in A, \|a\|=\|b\|=1\} \\ &= |\alpha|\|\varphi\|\|\psi\|\sup\{\|m(a,b)\| : a,b\in A, \|a\|=\|b\|=1\} \\ &= |\alpha||||m|||. \end{aligned} \tag{3.10}$$

For $m,m'\in \mathbf{Q}M_{(\varphi,\psi)}(A)$ and $\varepsilon>0$, since

$$\frac{|||m+m'|||}{\|\varphi\|\|\psi\|} = \sup\{\|(m+m')(a,b)\| : a,b\in A, \|a\|=\|b\|=1\},$$

choose $a,b\in A$ such that $\|a\|=\|b\|=1$ and $\|(m+m')(a,b)\| > \frac{|||m+m'|||}{\|\varphi\|\|\psi\|} - \varepsilon$. Then

$$\frac{|||m|||+|||m'|||}{\|\varphi\|\|\psi\|} = \frac{|||m|||}{\|\varphi\|\|\psi\|} + \frac{|||m'|||}{\|\varphi\|\|\psi\|} \geq \|m(a,b)\| + \|m'(a,b)\|$$

$$\geq \quad \|(m+m')(a,b)\| > \frac{|||m+m'|||}{\|\varphi\|\|\psi\|} - \varepsilon,$$

since $\varepsilon > 0$ was arbitrary then

$$|||m||| + |||m'||| > |||m+m'|||. \tag{3.11}$$

Hence by (3.9), (3.10) and (3.11), $|||\cdot|||$ is a norm on $\mathbf{Q}M_{(\varphi,\psi)}(A)$.

Now, let $\{m_n\}$ be a Cauchy sequence in $\mathbf{Q}M_{(\varphi,\psi)}(A)$. Then

$$\begin{aligned} \frac{1}{\|a\|\|b\|}\|m_n(a,b) - m_t(a,b)\| &= \|m_n(\frac{a}{\|a\|}, \frac{b}{\|b\|}) - m_t(\frac{a}{\|a\|}, \frac{b}{\|b\|})\| \\ &= \|(m_n - m_t)(\frac{a}{\|a\|}, \frac{b}{\|b\|})\| \\ &\leq \frac{1}{\|\varphi\|\|\psi\|}|||m_n - m_t|||, \end{aligned}$$

for every none zero $a, b \in A$. Thus

$$\limsup_{n,t} \|m_n(a,b) - m_t(a,b)\| \leq \limsup_{n,t} \frac{\|a\|\|b\|}{\|\varphi\|\|\psi\|}||||m_n - m_t||| = 0, \tag{3.12}$$

for every none zero $a, b \in A$. Define $m : A \times A \longrightarrow A$ with $m(a,b) = \lim_n m_n(a,b)$, for every $a, b \in A$. A simple limit argument shows that m is in $\mathbf{Q}M_{(\varphi,\psi)}(A)$.

Let ε be a positive number, since $\{m_n\}$ is a Cauchy sequence in $\mathbf{Q}M_{(\varphi,\psi)}(A)$, so there exists $N \in \mathbb{N}$ such that for all $s, t \in \mathbb{N}$ that $s, t > N$, we have $|||m_s - m_t||| < \frac{\varepsilon}{2}$. Let a, b be elements of A such that $\|a\| = \|b\| = 1$. Since $\lim_t m_t(a,b) = m(a,b)$, then for all $t \in \mathbb{N}$ such that $t > N$, we have $\|m_t(a,b) - m(a,b)\| < \frac{\varepsilon}{2}$. Then for all $s > N$,

$$\begin{aligned} \|\varphi\|\|\psi\|\|m_s(a,b) - m(a,b)\| &\leq \|\varphi\|\|\psi\|\|m_s(a,b) - m_t(a,b)\| \\ &+ \|\varphi\|\|\psi\|\|m_t(a,b) - m(a,b)\| \\ &< |||m_s - m_t||| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{3.13}$$

Hence $\mathbf{Q}M_{(\varphi,\psi)}(A)$ is complete. Therefore $\mathbf{Q}M_{(\varphi,\psi)}(A)$ is a Banach space. □

Corollary 3.7. *Let A be a Banach algebra with bounded approximate identity and $\varphi, \psi : A \longrightarrow A$ be linear maps, $\|\varphi\|, \|\psi\| \leq 1$. Then $(\mathbf{Q}M_{(\varphi,\psi)}(A), |||\cdot|||)$ is a Banach space.*

Proposition 3.8. *Let A be a Banach algebra. Then $(\mathbf{Q}M_{(\varphi,\psi)}(A), |||\cdot|||)$ is a Banach A-bimodule.*

Proof. Theorem 3.6, follows that $(\mathbf{Q}M_{(\varphi,\psi)}(A), |||\cdot|||)$ is a Banach space. Let $m \in \mathbf{Q}M_{(\varphi,\psi)}(A)$ and a arbitrary element in A. Now, we define left and right module actions as follows:

$$(a\cdot m)(x,y) = m(xa,y), \qquad \text{and} \qquad (m\cdot a)(x,y) = m(x,ay),$$

for every $x,y \in A$. We have to show that $a\cdot m$ and $m\cdot a$ belong to $\mathbf{Q}M_{(\varphi,\psi)}(A)$. Then for every $x_1, x_2, y_1, y_2 \in A$ we have

$$\begin{aligned}(a\cdot m)(x_1x_2, y_1y_2) &= m(x_1x_2\cdot a, y_1y_2) = \varphi(x_1)m(x_2\cdot a, y_1)\psi(y_2) \\ &= \varphi(x_1)((a\cdot m)(x_2,y_1))\psi(y_2), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}(m\cdot a)(x_1x_2, y_1y_2) &= m(x_1x_2, a\cdot y_1y_2) = \varphi(x_1)m(x_2, a\cdot y_1)\psi(y_2) \\ &= \varphi(x_1)((m\cdot a)(x_2,y_1))\psi(y_2). \end{aligned} \tag{3.15}$$

Thus, (3.14) and (3.15) yield that $a\cdot m$ and $m\cdot a$ are (φ,ψ)-quasi multipliers. The boundedness of $a\cdot m$ and $m\cdot a$ obtained by

$$\|(a\cdot m)(x,y)\| = \|m(xa,y)\| \leq |||m|||\|x\|\|a\|\|y\|,$$

and $\|(m\cdot a)(x,y)\| = \|m(x,ay)\| \leq |||m|||\|x\|\|a\|\|y\|$. □

If $\varphi,\psi : A \longrightarrow A$ be none zero linear maps, for each $a \in A$, we define $L : A \longrightarrow \mathbf{Q}M_{(\varphi,\psi)}(A)$ by, $L(a)(x,y) = \frac{\varphi(x)a\psi(y)}{\|\varphi\|\|\psi\|}$, for all $x,y \in A$.

Lemma 3.9. *Let A be a Banach algebra, let $\{x_\alpha\}$ be a bounded approximate identity for A and let $\varphi,\psi : A \longrightarrow A$ be bounded and surjective homomorphisms. Then*

$$\lim_\alpha \|\varphi(x_\alpha)a\psi(x_\alpha) - a\| = 0,$$

for all $a \in A$.

Proof. Choose $x,y \in A$ such that $a = xy$. Since φ and ψ are onto, so there exist $y_1, b \in A$ such that $y = \psi(y_1)$ and $a = \varphi(b)$. Then

$$\lim_\alpha \|\varphi(x_\alpha)a\psi(x_\alpha) - a\| = \lim_\alpha \|\varphi(x_\alpha)xy\psi(x_\alpha) - \varphi(x_\alpha)xy + \varphi(x_\alpha)xy - xy\|$$

$$
\begin{aligned}
&\leq \lim_\alpha \|\varphi(x_\alpha)x\|\|y\psi(x_\alpha)-y\| + \lim_\alpha \|\varphi(x_\alpha)xy - xy\| \\
&\leq \lim_\alpha \|\varphi(x_\alpha)\|\|x\|\|y\psi(x_\alpha)-y\| + \lim_\alpha \|\varphi(x_\alpha)a - a\| \\
&\leq \lim_\alpha \|\varphi\|\|x_\alpha\|\|x\|\|\psi(y_1)\psi(x_\alpha) - \psi(y_1)\| \\
&\quad + \lim_\alpha \|\varphi(x_\alpha)\varphi(b) - \varphi(b)\| \\
&\leq \lim_\alpha \|\varphi\|\|x_\alpha\|\|x\|\|\psi(y_1 x_\alpha - y_1)\| + \lim_\alpha \|\varphi(x_\alpha b - b)\| \\
&\leq \lim_\alpha CDMN\|y_1 x_\alpha - y_1\| + \lim_\alpha M\|x_\alpha b - b\| \\
&= 0,
\end{aligned}
$$

where, C, D, M and N are bound for $\{x_\alpha\}$, x, φ and ψ, respectively. □

Theorem 3.10. *Let A be a Banach algebra and let $\{x_\alpha\}$ be a bounded approximate identity for A, then the map L is linear isometry of A into $\mathbf{Q}M_{(\varphi,\psi)}(A)$, that $\varphi,\psi : A \longrightarrow A$ are none zero, with bound 1 and surjective homomorphisms.*

Proof. Linearity of L is trivial. For every $a \in A$ we have

$$
\begin{aligned}
|||L(a)||| &= \|\varphi\|\|\psi\| \sup\{\|L(a)(x,y)\| : x, y \in A, \|x\| = \|y\| = 1\} \\
&= \|\varphi\|\|\psi\| \sup\{\frac{\|\varphi(x)a\psi(y)\|}{\|\varphi\|\|\psi\|} : x, y \in A, \|x\| = \|y\| = 1\} \\
&\leq \sup\{\|\varphi(x)\|\|a\|\|\psi(y)\| : x, y \in A, \|x\| = \|y\| = 1\} \\
&\leq \sup\{\|\varphi\|\|x\|\|a\|\|\psi\|\|y\| : x, y \in A, \|x\| = \|y\| = 1\} \\
&\leq \|a\|. \qquad (3.16)
\end{aligned}
$$

Lemma 3.9 implies

$$
\begin{aligned}
|||L(a)||| &= \|\varphi\|\|\psi\| \sup\{\|L(a)(x,y)\| : x, y \in A, \|x\| = \|y\| = 1\} \\
&\geq \|\varphi\|\|\psi\| \limsup_\alpha \|L(a)(x_\alpha, x_\alpha)\| \\
&= \limsup_\alpha \|\varphi(x_\alpha)a\psi(x_\alpha)\| \\
&= \|a\|. \qquad (3.17)
\end{aligned}
$$

for all $a \in A$. Hence, (3.16) and (3.17) imply $|||L(a)||| = \|a\|$, for all $a \in A$. This means that L is an isometry from A into $\mathbf{Q}M_{(\varphi,\psi)}(A)$. □

By applying Corollary 3.5, Lemma 3.9 and similar method in above stated Theorem proof, one can show that:

Lemma 3.11. *Let A be a Banach algebra with bounded approximate identity, and let $\varphi, \psi : A \longrightarrow A$ be bounded with 1 and surjective homomorphisms. Then*

(i) $\lambda : M_{\psi l}(A) \longrightarrow \boldsymbol{Q}M_{(\varphi,\psi)}(A)$ *with definition* $[\lambda(T)](a,b) = \varphi(a)T(b)$, *for all* $a \in A$ *and* $T \in M_{\varphi r}(A)$, *is isometry.*

(ii) $\rho : M_{\varphi r}(A) \longrightarrow \boldsymbol{Q}M_{(\varphi,\psi)}(A)$ *with definition* $[\rho(T)](a,b) = T(a)\psi(b)$, *for all* $a \in A$ *and* $T \in M_{l\varphi}(A)$, *is isometry.*

A bounded approximate identity $(e_\alpha)_{\alpha\in I}$ in Banach algebra A is said to be ultra-approximate identity if, for all $m \in \mathbf{Q}M_{(\varphi,\psi)}(A)$ and $a \in A$ the nets $(m(e_\alpha, a))_{\alpha\in I}$ and $(m(a, e_\alpha))_{\alpha\in I}$ are Cauchy in A.

Theorem 3.12. *Let A be a Banach algebra with ultra-approximate identity and φ, ψ be homomorphisms from A to A. Then*

(i) the mapping $\lambda : M_{\psi l}(A) \longrightarrow \boldsymbol{Q}M_{(\varphi,\psi)}(A)$ *with definition* $[\lambda(T)](a,b) = \varphi(a)T(b)$, *for all* $a \in A$, *is surjective.*

(ii) the mapping $\rho : M_{\varphi r}(A) \longrightarrow \boldsymbol{Q}M_{(\varphi,\psi)}(A)$ *with definition* $[\rho(T)](a,b) = T(a)\psi(b)$, *for all* $a \in A$, *is surjective.*

(iii) if $\psi \circ \varphi = id_A$, ψ *and* φ *are bounded and* ψ *is idempotent, then*

$$(\lambda^{-1} \circ \rho(f)) \circ (\lambda^{-1} \circ \rho(g)) = \lambda^{-1} \circ \rho(\psi(g \circ f)).$$

(iv) if $\varphi \circ \psi = id_A$, ψ *and* φ *are bounded and* φ *is idempotent, then*

$$(\rho^{-1} \circ \lambda) \circ (\rho^{-1} \circ \lambda) = \rho^{-1} \circ \lambda(\varphi(g \circ f)).$$

Proof. Let $(e_\alpha)_{\alpha\in I}$ be an ultra-approximate identity. Let m be in $\mathbf{Q}M_{(\varphi,\psi)}(A)$.

(i) Define $f : A \longrightarrow A$ by letting, $f(a) = \lim_\alpha m(e_\alpha, a)$, for each $a \in A$. Then

$$f(ab) = \lim_\alpha m(e_\alpha, ab) = \lim_\alpha m(e_\alpha, a)\psi(b) = f(a)\psi(b),$$

for all $a, b \in A$. It follows that $f \in M_{\psi l}(A)$. For all $a, b \in A$,

$$[\lambda(f)](a,b) = \varphi(a)f(b) = \lim_\alpha \varphi(a)m(e_\alpha, b) = \lim_\alpha m(ae_\alpha, b) = m(a,b).$$

Thus $\lambda(f) = m$ and therefore λ is surjective.

(ii) Define $f : A \longrightarrow A$ by $f(a) = \lim_\alpha m(a, e_\alpha)$, for each $a \in A$. Then

$$f(ab) = \lim_\alpha m(ab, e_\alpha) = \lim_\alpha \varphi(a)m(b, e_\alpha) = \varphi(a)f(b),$$

for each $a, b \in A$. It follows that $f \in M_{\varphi r}(A)$. Then

$$[\rho(f)](a,b) = f(a)\psi(b) = \lim_{\alpha} m(a, e_\alpha)\psi(b) = \lim_{\alpha} m(a, e_\alpha b) = m(a,b),$$

for all $a, b \in A$. Thus $\rho(f) = m$ and ρ is surjective.

(ii) Let $f, g \in M_{\varphi r}(A)$ such that f and g defined like as mapping f in case (i). Then for every $a \in A$ we have

$$\begin{aligned} \langle a, (\lambda^{-1} \circ \rho(f)) \circ (\lambda^{-1} \circ \rho(g))\rangle &= (\lambda^{-1} \circ \rho(f)) \circ (\lim_{\alpha}[\rho(g)](e_\alpha, a)) \\ &= (\lambda^{-1} \circ \rho(f))(\lim_{\alpha}[\rho(g)](e_\alpha, a)) \\ &= (\lambda^{-1} \circ \rho(f))(\lim_{\alpha} g(e_\alpha)\psi(a)) \\ &= \lim_{\beta} f(e_\beta) \lim_{\alpha} \psi(g(e_\alpha))\psi(\psi(a)) \\ &= \lim_{\beta} f(e_\beta) \lim_{\alpha} \psi(g(e_\alpha))\psi(a). \end{aligned}$$

Now, we claim that $\lim_\beta f(e_\beta) \lim_\alpha \psi(g(e_\alpha))\psi(a) = \lim_{\delta \in I} \psi(g \circ f)(e_\delta)\psi(a)$. Take $B = \lim_\alpha g(e_\alpha)\psi(a)$, $C = \lim_\beta f(e_\beta)\psi(B)$, and $D = \lim_\delta \psi(g \circ f)(e_\delta)\psi(a)$. Choose $\sigma, \gamma \in I$ such that $\|f(e_\sigma)\psi(B) - C\| < \varepsilon/4$, $\|\psi(g \circ f)(e_\sigma)\psi(a) - D\| < \varepsilon/4$, $\|g(e_\gamma)\psi(a) - B\| < \varepsilon/(4\|f\|\|\psi\|)$ and $\|f(e_\sigma)e_\gamma - f(e_\sigma)\| < \varepsilon/(\|\psi\|^2\|g\|\|a\|)$. Then

$$\begin{aligned} \|C - D\| \le{}& \|f(e_\sigma)\psi(B) - C\| + \|f(e_\sigma)\psi(g(e_\gamma))\psi(a) - f(e_\sigma)\psi(B)\| \\ &+ \|\psi(g(f(e_\sigma)e_\gamma))\psi(a) - \psi(g(f(e_\sigma)))\psi(a)\| + \|\psi(g \circ f)(e_\sigma)\psi(a) - D\| \\ <{}& \varepsilon/4 + \|f(e_\sigma)\|\|\psi\|\|g(e_\gamma)\psi(a) - B\| \\ &+ \|\psi\|\|g\|\|f(e_\sigma)e_\gamma - f(e_\sigma)\|\|\psi(a)\| + \varepsilon/4 \\ <{}& \varepsilon. \end{aligned}$$

Since $\lambda^{-1} \circ \rho(\psi(g \circ f))(a) = \lim_\alpha \psi(g \circ f)(e_\alpha)\psi(a)$, therefore (iii) hold.

(iv) Proof is similar to (iii). □

Let A be a Banach algebra. The second dual of A^{**} of A equipped with the first (second) Arens product $\Box$ ($\Diamond$) is a Banach algebra.

Theorem 3.13. *Let A be a factorable Banach algebra and $m \in \boldsymbol{Q}M_{(\varphi,\psi)}(A)$, such that $\varphi, \psi : A \longrightarrow A$ are bounded linear maps. Then m^{***} is continuous and belongs to $\boldsymbol{Q}M_{(\varphi^{**},\psi^{**})}(A^{**})$.*

Proof. Consider A^{**} with the first Arens product $\Box$. Continuity of φ and ψ imply continuity of m (Theorem 3.3). Then the first transpose of m is defined to be the continuous bilinear mapping $m^*: A^* \times A \longrightarrow A^*$ given by the following formula

$$\langle m^*(a^*, b), c\rangle := \langle a^*, m(b, c)\rangle \qquad (a^* \in A^*, b, c \in A).$$

We continue to define $m^{**}: A^{**} \times A^* \longrightarrow A^*$ and $m^{***}: A^{**} \times A^{**} \longrightarrow A^{**}$. The mapping m^{***} extends m in the sense that $m^{***}|_{A\times A} = m$. Clearly, the second transpose of φ and ψ are continuous linear maps from A^{**} into A^{**}. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in A^{**}$. Then Goldestin's Theorem implies that there are nets $(a_\alpha), (b_\beta), (c_\gamma), (d_\eta) \in A$ such that $a_\alpha \xrightarrow{\text{w}^*} \mathfrak{a}, b_\beta \xrightarrow{\text{w}^*} \mathfrak{b}, c_\gamma \xrightarrow{\text{w}^*} \mathfrak{c}$ and $d_\eta \xrightarrow{\text{w}^*} \mathfrak{d}$. Then

$$\begin{aligned} m(\mathfrak{a}\Box\mathfrak{b}, \mathfrak{c}\Box\mathfrak{d}) &= \text{w}^*-\lim_\alpha\ \text{w}^*-\lim_\beta\ \text{w}^*-\lim_\gamma\ \text{w}^*-\lim_\eta m(a_\alpha b_\beta, c_\gamma d_\eta) \\ &= \text{w}^*-\lim_\alpha\ \text{w}^*-\lim_\beta\ \text{w}^*-\lim_\gamma\ \text{w}^*-\lim_\eta \varphi(a_\alpha) m(b_\beta, c_\gamma)\psi(d_\eta) \\ &= \varphi^{**}(\mathfrak{a})\Box m^{**}(\mathfrak{b}, \mathfrak{c})\Box\psi^{**}(\mathfrak{d}). \end{aligned}$$

By the similar method, we have the similar result in case that A^{**} equipped with the second Arens product $\diamondsuit$. $\Box$

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