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COLOR CLASS DOMINATION AND COLORFUL DOMINATION

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Abstract: Let $G = (V, E)$ be a finite, simple and undirected graph. A partition of $V(G)$ into independent sets such that each independent set is dominated by a vertex belonging to V is called a color class domination partition (in short *cd*-partition) [[3], [4], [10], [11], [12]]. The minimum cardinality of a *cd*-partition is called the *cd*-chromatic number of G and is denoted by $\chi_{cd}(G)$. A proper coloring of G in which each vertex of the graph dominates some color class is called a dominator coloring of G and the minimum number of color classes in a dominator coloring of G is called the dominator chromatic number of G and is denoted by $\chi_d(G)$ [[1], [5],[6], [8]]. In a χ_d-partition of G, any set formed by selecting one vertex each from every color class becomes a dominating set of G. That is, in such a dominating set each vertex has distinct color and no color is represented by more than one element. Thus, a χ_d-coloring gives rise to a dominating set which can be rightly called a colorful dominating set of G (also called gamma coloring of G [8]). In the case of χ_{cd}-partition, this does not happen generally. So, it is a nice problem to find the minimum cardinality of a *cd*-partition which will

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give rise to a colorful dominating set. This new parameter is denoted by $\chi_{\gamma}^{cd}(G)$. A study of this parameter is initiated in this paper.

AMS Subject Classification: 05C69
Key Words: color class domination partition, dominator chromatic number, colorful dominating set, dominator coloring

1. Introduction

Gera et al introduced the concept of dominator coloring [[5], [6]]. A proper color partition of $V(G)$ is called a dominator color partition if every vertex of G dominates some color class. The minimum cardinality of a dominator coloring is called the dominator chromatic number of G and is denoted by $\chi_d(G)$. In a χ_d-partition of G, any set formed by selecting one vertex each from every color class becomes a dominating set of G. Another type of proper color partition is the color class domination partition in which each color class is dominated by a vertex of G [[3], [4], [10], [11], [12]]. The minimum cardinality of a *cd*-partition of G is denoted by $\chi_{cd}(G)$. In the case of a χ_{cd}-partition, the selection of one vertex from each color class need not form a dominating set of G. Hence, the problem arises of finding the minimum cardinality of a *cd*-partition of G such that the set formed by taking one vertex from each color class is a dominating set. The minimum cardinality of such set is denoted by $\chi_{\gamma}^{cd}(G)$.

Definition 1.1 ([5],[6]). *Let $G = (V, E)$ be a simple, finite and undirected graph. A proper color partition of G is called a* **dominator coloring** *of G if each vertex of G dominates a color class. The minimum cardinality of such a partition is called the* **dominator chromatic number** *of G and is denoted by $\chi_d(G)$.*

Definition 1.2 ([3],[4]). *A proper color partition of G is called a* **color class domination partition** *of G (cd-partition of G) if every color class is dominated by a vertex of G. The minimum cardinality of such a partition is called the cd-***chromatic number** *of G and is denoted by $\chi_{cd}(G)$.*

Remark 1.3 ([3],[4]). *There is no relationship between $\chi_d(G)$ and $\chi_{cd}(G)$.*

2. Colorful Dominating set

Definition 2.1. *Let G be a finite, simple and undirected graph. Let $\Pi = \{V_1, V_2, \ldots, V_r\}$ be a color class domination partition of G. Suppose $S = \{x_1, x_2, \ldots, x_r\}$ be a subset of $V(G)$ such that $x_i \in V_i$, $(1 \leq i \leq r)$ and S is a dominating set of G. Then S is called a* **colorful dominating set** *with respect to* Π.

Definition 2.2. *A Color class domination partition Π of a graph G is called a* **Colorful dominating color class domination partition** *of G if Π admits a colorful dominating set and in such a case G has a colorful dominating set with respect to cd-partition.*

Remark 2.3. *For any graph G, trivial coloring serves as a colorful dominating color class domination partition.*

Definition 2.4. *The minimum cardinality of a colorful dominating set with respect to all color class domination partitions of G is called the* **colorful domination number** *of G with respect to color class domination partition of G and is denoted by $\chi_\gamma^{cd}(G)$.*

3. $\chi_\gamma^{cd}(G)$ for Well Known Graphs

(1). $\chi_\gamma^{cd}(K_n) = n = \chi(K_n)$ where as $\gamma(K_n) = 1$.
(2). $\chi_\gamma^{cd}(D_{r,s}) = 2 = \chi(D_{r,s}) = \chi_{cd}(D_{r,s})$.
(3). $\chi_\gamma^{cd}(K_{1,n}) = 2 = \chi(K_{1,n})$ where as $\gamma(K_{1,n}) = 1$.
(4). $\chi_\gamma^{cd}(K_{m,n}) = 2 = \chi(K_{m,n})$ where as $\gamma(K_{m,n}) = 2$.
(5). $\chi_\gamma^{cd}(P_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2(\bmod 4) \\ \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0,\ 1,\ 3(\bmod 4) \end{cases}$

Proof. When $n = 4k+2$, $\chi_{cd}(P_n)$-partition is $\{\{1,3\},\{2,4\},\{5,7\},\{6,8\}, \ldots, \{4k-7, 4k-5\}, \{4k-6, 4k-4\}, \{4k-3, 4k-1\}, \{4k-2, 4k\}, \{4k+1\}, \{4k+2\}\}$. Therefore $\chi_{cd}(P_{4k+2}) = 2k+2 = \frac{n}{2} + 1$ where $n = 4k+2$.

When $n = 4k+1$, $\chi_{cd}(P_n)$-partition is $\{\{1,3\},\{2,4\},\{5,7\},\{6,8\}, \ldots, \{4k-7, 4k-5\}, \{4k-6, 4k-4\}, \{4k-3, 4k-1\}, \{4k-2, 4k\}, \{4k+1\}\}$. Therefore $\chi_{cd}(P_{4k+1}) = 2k+1 = \lceil \frac{n}{2} \rceil$ where $n = 4k+1$.

When $n = 4k+3$, $\chi_{cd}(P_n)$-partition is $\{\{1,3\},\{2,4\},\{5,7\},\{6,8\}, \ldots, \{4k-7, 4k-5\}, \{4k-6, 4k-4\}, \{4k-3, 4k-1\}, \{4k-2, 4k\}, \{4k+1\}, \{4k+3\}\}$. Therefore $\chi_{cd}(P_{4k+3}) = 2k+2 = \lceil \frac{4k+3}{2} \rceil = \lceil \frac{n}{2} \rceil$ where $n = 4k+3$.

When $n = 4k$, $\chi_{cd}(P_n)$-partition is $\{\{1,3\},\{2,4\},\{5,7\},\{6,8\},\ldots,\{4k-7,4k-5\},\{4k-6,4k-4\},\{4k-3,4k-1\},\{4k-2,4k\}$. Therefore $\chi_{cd}(P_{4k}) = 2k = \lceil \frac{4k}{2} \rceil = \lceil \frac{n}{2} \rceil$ where $n = 4k$. □

(6). $\chi_{cd}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0,\ 1,\ 3 (\text{mod } 4),\ n \geq 4 \\ \frac{n}{2}+1 & \text{if } n \equiv 2(\text{ mod } 4) \\ 3 & \text{if } n = 3 \end{cases}$

χ_{cd}- partitions are the same as in P_n.

(7). $\chi_\gamma^{cd}(C_n) = \chi_{cd}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0,\ 1,\ 3(\text{ mod } 4),\ n \geq 4 \\ \frac{n}{2}+1 & \text{if } n \equiv 2(\text{ mod } 4) \\ 3 & \text{if } n = 3 \end{cases}$

(8). $\chi_{cd}(W_n) = \gamma(W_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even} \end{cases}$

Proof. Let $n = 2k+1$, $k \geq 2$. Then $\{\{1\},\{2,4,6,\ldots,2k\},\{3,5,7,\ldots,2k+1\}\}$ is a χ_{cd}-partition of W_n. When $n = 2k$, $k \geq 2$. Then $\{\{1\},\{2,4,6,\ldots,2k-2\},\{3,5,7,\ldots,2k-1,2k\}\}$ is a χ_{cd}-partition of W_n. □

(9). $\chi_\gamma^{cd}(W_n) = \chi_{cd}(W_n) = \begin{cases} 3 & \text{if } n = 2k+1,\ k \geq 2 \\ 4 & \text{if } n = 2k,\ k \geq 2 \end{cases}$

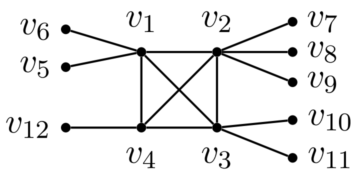
(10). Let $G = K_{n_1,n_2,n_3,\ldots,n_r}$. Let $V_1, V_2, \ldots, V_r$ be r-partite set of G. $|V_i| = n_i$, $(1 \leq i \leq r)$, $\Pi = \{V_1, V_2, \ldots, V_r\}$. Then Π is a χ_{cd}-partition of G. Let $x_i \in V_i$, $(1 \leq i \leq r)$. Then $D = \{x_1, x_2, \ldots, x_r\}$ is clearly a dominating set of G. $\chi_\gamma(G) = r = \chi_{cd}(G)$.

Definition 3.1 (Multi-Star Graph). *A multi-star graph $K_m(a_1, a_2, \ldots, a_m)$ is formed by taking a complete graph K_m with vertex set $\{x_1, x_2, \ldots, x_r\}$ and joining a_i pendent vertices $(a_i \geq 1,\ 1 \leq i \leq m)$ at x_i, $(1 \leq i \leq m)$.*

(11). $\chi_{cd}(G) = \chi_\gamma^{cd}(G)$, where G is a multi-star graph.

Proof. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_{k_i}}$ be the pendants attached at x_i, $(1 \leq i \leq m)$. Let $\Pi = \{\{v_1, v_{2_1}, \ldots, v_{2_{k_2}}\},\{v_2, v_{3_1}, \ldots, v_{3_{k_3}}\},\ldots,\{v_m, v_{1_1}, \ldots, v_{1_{k_1}}\}\}$ is a χ_{cd}-partition of cardinality m. $\{v_1, v_2, \ldots, v_m\}$ is a colorful dominating set of G with respect to this χ_{cd}-partition. Thus $\chi_{cd}(G) = \chi_\gamma^{cd}(G)$. □

Illustration 3.2. *Let G be the graph given below. Let $\Pi = \{\{v_1, v_7, v_8, v_9\}, \{v_2, v_5, v_6\}, \{v_3, v_{12}\}, \{v_4, v_{10}, v_{11}\}\}$. $\chi_{cd}(G) = 4$. $\chi_\gamma^{cd}(G) = 4$. $D = \{v_1, v_2, v_3, v_4\}$ is a dominating set of G.*



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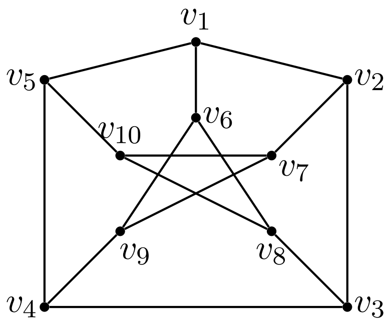
(12). $\chi_\gamma^{cd}(P) = 4 = \chi_{cd}(P)$ where P is the Petersen graph.

Proof. $\Pi = \{\{v_1, v_3, v_7\}, \{v_2, v_4, v_8\}, \{v_5, v_6\}, \{v_9, v_{10}\}\}$ is a χ_{cd}-partition of P. $\{v_1, v_4, v_6, v_{10}\}$ is a colorful dominating set of P. □

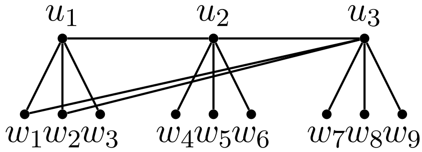


Petersen Graph

4. Bounds

Remark 4.1. $max\{\chi(G), \gamma(G)\} \leq max\{\chi_{cd}(G), \gamma(G)\} \leq \chi_\gamma^{cd}(G)$.

Example 4.2. *Let G be the graph given below:*



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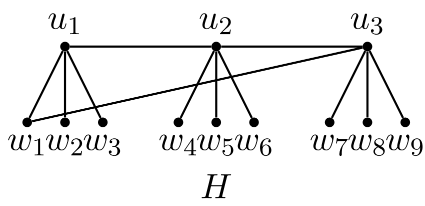
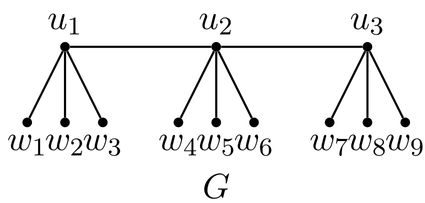
Then $\chi_{cd}(G) = 3$ *and* $\Pi = \{\{u_1, u_3, w_4, w_5, w_6\}, \{u_2, w_1, w_2, w_7, w_8, w_9\}, \{w_3\}\}$ *is a* χ_{cd}*-partition of* G. $S = \{u_3, u_2, w_3\}$ *is a colorful dominating set of* G. *Since* $\gamma(G) = 3$, S *is a* $\chi_\gamma^{cd}(G)$*-dominating set of* G. *That is,* $\chi_\gamma^{cd}(G) = \chi_{cd}(G) = 3$.

Example 4.3. *Let* H *be the graph given below:*



$\Pi = \{\{u_1, u_3, w_4, w_5, w_6\}, \{u_2, w_1, w_7, w_8, w_9\}, \{w_2, w_3\}\}$ *is a* χ_{cd}*-partition of* G. $\Pi_1 = \{\{u_1, u_3, w_4, w_5, w_6\}, \{u_2, w_1, w_2, w_3\}, \{w_7, w_8, w_9\}\}$ *is also a* χ_{cd}*-partition of* G. *In both partitions, selection of one element each from every color class do not constitute a dominating set. Let* $\Pi_2 = \{\{u_1, w_4, w_5, w_6\}, \{u_2, w_1, w_2, w_3\}, \{w_7, w_8, w_9\}, \{u_3\}\}$. Π_2 *is a colorful dominating color class domination partition of* H. *Therefore* $\chi_\gamma^{cd}(G) = 4 > \chi_{cd}(H)$.

Example 4.4. *Let* G *be the graph given below:*



Then $\Pi_1 = \{\{u_1, u_3, w_4, w_5, w_6\}, \{u_2, w_1, w_7, w_8, w_9\}, \{w_2, w_3\}\}$ *is a* χ_{cd}*-partition of* G. $\Pi_2 = \{\{u_1, u_3, w_4, w_5, w_6\}, \{u_2, w_1, w_2, w_3\}, \{w_7, w_8, w_9\}\}$ *is also a* χ_{cd}*-partition of* G.

Suppose there exists a χ_{cd}*-partition of* G *such that* $\Pi_3 = \{V_1, V_2, V_3\}$ *where* $u_1 \in V_1$, $u_2 \in V_2$, $u_3 \in V_3$. *Then* $V_1 = \{u_1, w_4, w_5, w_6\}$, $V_2 = \{u_2, w_1, w_2, w_3\}$ *or* $\{u_2, w_1, w_7, w_8, w_9\}$ *and* $V_3 = \{u_3\}$. *Hence either* $V_1 \cup V_2 \cup V_3 = V - \{w_2, w_3\}$ *or* $V - \{w_7, w_8, w_9\}$. *Therefore* Π_3 *is not a partition of* $V(G)$. *Therefore there exists no* χ_{cd}*-partition of* G *such that it gives rise to* χ_γ^{cd}*-partition. Therefore* $\chi_\gamma^{cd}(G) > \chi_{cd}(G) = 3$. *Suppose* $V_1 = \{u_1, w_4, w_5, w_6\}$, $V_2 = \{w_2, w_3\}$, $V_3 = \{u_2, w_1, w_7, w_8, w_9\}$ *and* $V_4 = \{u_3\}$. *Here* $\Pi' = \{V_1, V_2, V_3, V_4\}$ *is a cd-partition of* G *and is also a* $\Pi_\gamma^{cd}(G)$*-partition of* G. *Therefore* $\chi_\gamma^{cd}(G) = 4$.

Proposition 4.5. *For any graph G on n vertices, $1 \leq \chi_\gamma^{cd}(G) \leq n$.*

Proof. Since $\chi_\gamma^{cd}(G)$ is the cardinality of a dominating set of G, $1 \leq \chi_\gamma^{cd}(G) \leq n$. □

Observation 4.6. $\chi_\gamma^{cd}(G) = 1$ *iff* $G = K_1$.

Theorem 4.7. $\chi_\gamma^{cd}(G) = n$ *iff* $G = K_n$ *or* $K_{r_1} \cup K_{r_2} \cup \ldots \cup K_{r_k}$.

Proof. Let $\chi_\gamma^{cd}(G) = n$.
Case 1: Let G be connected.

Suppose G has two or more independent vertices. Let x, $y \in V(G)$ be independent. Since G is connected, there exists $z \in V(G)$ such that x and z are independent and have a common adjacent vertex. Therefore $\Pi = \{\{x, z\}, V_2, V_3, \ldots, V_{n-1}\}$ where each V_i, $2 \leq i \leq n-1$, is a singleton say u_i. $S = \{x, u_2, \ldots, u_{n-1}\}$ is a colorful dominating set of G. Therefore

$$\chi_\gamma^{cd}(G) \leq n - 1,$$

a contradiction. Therefore any two vertices in G are adjacent. Hence $G = K_n$.
Case 2: Let G be disconnected.

Let $G_1, G_2, \ldots, G_k$ be the components of G. Therefore

$$\chi_\gamma^{cd}(G) = \sum_{i=1}^{k} \chi_\gamma^{cd}(G_i) = |V(G)|.$$

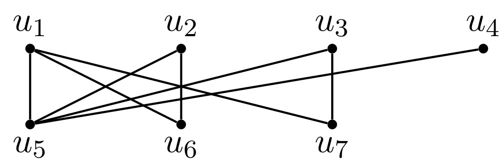
Therefore $\chi_\gamma^{cd}(G_i) = |V(G_i)|$, $1 \leq i \leq k$. Since G_i is connected, by case 1, each G_i is complete. That is, $G = K_{r_1} \cup K_{r_2} \cup \ldots \cup K_{r_k}$.
The converse is obvious □

Remark 4.8. (i). *If each $K_{r_i} = K_1$, then we get $G = \overline{K_k}$.*
(ii). *If each $G_i = K_2$, then $G = kK_2$.*

Theorem 4.9. $\chi_\gamma^{cd}(G) = 2$ *iff G is either K_2 or $\overline{K_2}$ or a bipartite graph with bipartition V_1, V_2 such that there exists $x_1 \in V_1$ which is adjacent with every vertex of V_2 and there exists $y_1 \in V_2$ which is adjacent with every vertex of V_1.*

Proof. If G is of the type given in the theorem, then $\chi_\gamma^{cd}(G) = 2$. Conversely, let $\chi_\gamma^{cd}(G) = 2$. Then there exists a *cd*-partition of cardinality 2. Let $\Pi = \{V_1, V_2\}$ be a *cd*-partition of G. If G is a null graph, then there is no edge between V_1 and V_2. Since $\chi_\gamma^{cd}(G) = 2$, there exist $x \in V_1$ and $y \in V_2$ such that $\{x, y\}$ is a dominating set of G. Since G is a null graph, $|V_1| = |V_2| = 1$. Therefore $G = \overline{K_2}$. Let G be a non-null graph. Let $\Pi = \{V_1, V_2\}$ be a *cd*-partition of G. Since $\chi_\gamma^{cd}(G) = 2$, there exist $x \in V_1$ and $y \in V_2$ such that $\{x, y\}$ is a dominating set of G. Therefore x is adjacent with every vertex of V_2 and y is adjacent with every vertex of V_1. Hence the theorem. □

Illustration 4.10. *Let G be the graph given below:*



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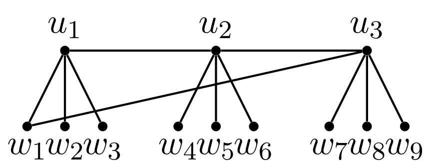
Let $\Pi = \{\{u_1, u_2, u_3, u_4\}, \{u_5, u_6, u_7\}\}$. $D = \{u_1, u_5\}$ is a dominating set of G.

Remark 4.11. *Let $\chi_\gamma^{cd}(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be a cd-partition of G such that $x \in V_1$ and $y \in V_2$ and $\{x, y\}$ is a dominating set of G. Therefore $\chi_{cd} \leq 2$. Suppose $\chi_{cd}(G) = 1$.Then $G = K_1$. Therefore $\chi_\gamma^{cd}(G) = 1$, a contradiction. Therefore $\chi_{cd} = 2$ and Π is a χ_{cd}-partition of G.*

Remark 4.12. *There is no relationship between $\chi_\gamma^{cd}(G)$ and $\chi_d(G)$. For: (i). $\chi_\gamma^{cd}(C_{20}) = 10$, $\chi_d(C_{20}) = 9$. Therefore $\chi_\gamma^{cd}(C_{20}) > \chi_d(C_{20})$.*
(ii). $\chi_\gamma^{cd}(C_{12}) = 6 = \chi_d(C_{12})$.
(iii). $\chi_\gamma^{cd}(D_(r, s) = 2 < \chi_d(D_{r,s}) = 3$, where $r, s \geq 2$.

Observation 4.13. *Given a positive integer k, there exists a connected graph G such that $\chi_\gamma^{cd}(G) - \chi_{cd}(G) = k$.*

Proof. Let H be the graph given below:



H

Take k-vertex disjoints copies of H. Join one pendant vertex of a copy of H with a pendant vertex of subsequent copy of H. Let G be the resulting graph. Then $\chi_\gamma^{cd}(G) = 4k$, $\chi_{cd}(G) = 3k$. Therefore $\chi_\gamma^{cd}(G) - \chi_{cd}(G) = k$. □

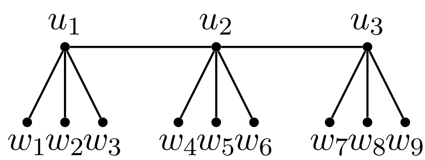
Theorem 4.14. *Let G be a simple graph. Then $\chi_\gamma^{cd}(G) \leq \chi_{cd}(G)+\gamma(G)-1$.*

Proof. Let $\Pi = \{V_1, V_2, \ldots, V_r\}$ be a χ_{cd}-partition of G where $r = \chi_{cd}(G)$. Let $D = \{x_1, x_2, \ldots, x_\gamma\}$ be a minimum dominating set of G. Assign $\chi_{cd}(G)+1, \chi_{cd}(G)+2, \ldots, \chi_{cd}(G)+\gamma(G)-1$ colors to the vertices $x_1, x_2, \ldots, x_{\gamma-1}$ leaving the other vertices colored as before. Let $\Pi_1 = \{V_1-(D-\{x_\gamma\}), V_2-(D-\{x_\gamma\}), \ldots, V_r-(D-\{x_\gamma\}), \{x_1\}, \{x_2\}, \ldots, \{x_{\gamma-1}\}$. $\Pi_1 = \{V_1-D_1, V_2-D_1, \ldots, V_r-D_1, \{x_1\}, \{x_2\}, \ldots, \{x_{\gamma-1}\}\}$ where $D_1 = \{x_1, x_2, \ldots, x_{\gamma-1}\}$. Since Π is a *cd*-partition of G, each V_i, $(1 \leq i \leq r)$ is dominated by a vertex y_i of G and hence each $V_i - D_1$ is dominated by y_i of G. Therefore Π_1 is a *cd*-partition of G. $|\Pi_1| = r+\gamma-1 = \chi_{cd}(G)+\gamma(G)-1$. Let $S = \{z_1, z_2, \ldots, z_r, x_1, x_2, \ldots, x_{\gamma-1}\}$ where $z_i \in (V_i - D)$, $(1 \leq i \leq r)$. Since $V_1 - D_1 \cup \ldots \cup V_r - D_1 = V - D_1$ and $x_\gamma \in V - D_1$, $x_\gamma \in V_j - D_1$, for some j, $(1 \leq j \leq r)$. Take $z_j = x_\gamma$. Then S is a dominating set of G. Therefore $\chi_\gamma^{cd}(G) \leq |S| = \chi_{cd}(G)+\gamma(G)-1$. □

Remark 4.15. *The bounds are sharp. For $\chi_\gamma^{cd}(K_n) = n = \chi_{cd}(K_n)$. $\gamma(K_n) = 1$. Therefore $\chi_\gamma^{cd}(G) = \chi_{cd}(G)+\gamma(G)-1$. Also $max\{\chi_{cd}(K_n), \gamma(K_n)\} = max\{n, 1\} = n = \chi_\gamma^{cd}(K_n)$.*

Remark 4.16. *There exist a connected graph G such that $\chi_\gamma^{cd}(G) \leq \chi_{cd}(G)+\gamma(G)-2$.*

Let G be the graph given below:



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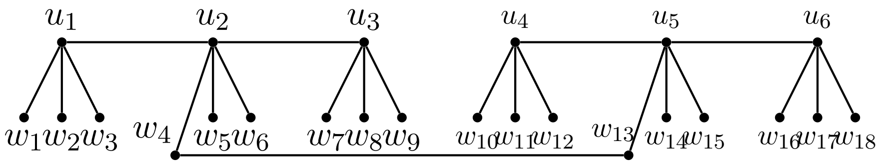
Let $\Pi = \{V_1 = \{u_1, u_3, w_4, w_5, w_6\}, V_2 = \{u_2, w_1, w_7, w_8, w_9\}, V_3 = \{w_2, w_3\}\}$. Since G is bipartite, $G \neq K_2$, $G \neq \overline{K_2}$ and there exist no vertex in any partite set of G which is adjacent with every vertex of the other partite set of G. Therefore , $\chi_{cd}(G) > 2$. Π is a χ_{cd}-partition of G and $\chi_{cd}(G) = 3$. Let $D = \{u_1, u_2, u_3\}$. Then D is a minimum dominating set of G. Let $\Pi_1 = \{V_1 - D_1, V_2 - D_1, V_3 - D_1, \{u_1\}\}$ where $D_1 = \{u_1\}$. Π_1 is a cd-partition of G. $|\Pi| = 4 = 3+3-2 = \chi_{cd}(G)+\gamma(G)-2$. Let $S = \{z_1, z_2, z_3, u_1\}$ where $z_1 \in V_1 - D_1$, $z_2 \in V_1$, $z_3 \in V_3 - D_1$. $(V_1-D_1)\cup(V_2-D_1)\cup(V_3-D_1) = V - \{u_1\}$. Here $u_2 \in V_2 - D_1$, $u_3 \in V_1 - D_1$. Let $z_1 = u_3$, $z_2 = u_2$. Let $S = \{u_3, u_2, z_3, u_1\}$.

S is a dominating set of G. Therefore S is a colorful dominating set with respect to Π_1. Hence $\chi_\gamma^{cd}(G) \leq |S| = \chi_{cd}(G) + \gamma(G) - 2$.

Remark 4.17. *Let G be the graph given below:*



G

Let $\Pi = \{\{u_1, w_4, w_5, w_6, u_3\}, \{u_2, w_1, w_7, w_8, w_9\}, \{w_2, w_3\}, \{w_{10}, w_{16}, w_{17}, w_{18}, u_5\}, \{u_4, u_6, w_{13}, w_{14}, w_{15}\}, \{w_{11}, w_{12}\}\}$. Π is a cd-partition of G. Suppose $\chi_{cd}(G) = 5$. Then there exist a χ_{cd}-partition $\Pi_1 = \{V_1, V_2, V_3, V_4, V_5\}$. Since $\Delta(G) = 5, V_1, V_2, V_3, V_4$ each has five elements and V_5 has four elements. V_1 is dominated by u_2, V_2 is dominated by u_3, V_3 is dominated by u_5 and V_4 is dominated by u_6. Therefore $V_1 = \{u_1, w_4, w_5, w_6, u_3\}$, $V_2 = \{u_2, w_1, w_7, w_8, w_9\}$, $V_3 = \{u_4, u_6, w_{13}, w_{14}, w_{15}\}$, $V_4 = \{w_{10}, w_{16}, w_{17}, w_{18}, u_5\}$, $V_2 = \{w_2, w_3, w_{11}, w_{12}\}$. V_5 is not dominated by any vertex of G.Therefore $\chi_{cd}(G) > 5$. Therefore $\chi_{cd}(G) = 6$ and $\gamma(G) = 6$. Therefore in a Π_γ^{cd}-partition of G, $u_1, u_2, u_3, u_4, u_5, u_6$, must appear in different sets of the partition. Since any χ_{cd}-partition contains two of u_1 to u_6 in one set and two of u_1 to u_6 in another set, no χ_{cd}-partition is a Π_γ^{cd}-partition. Thus, any χ_{cd}-partition has to be enlarged by two or more sets to get a Π_γ^{cd}-partition. $\Pi_\gamma^{cd}(G) = 8$.

5. For Bipartite Graphs

Remark 5.1. *(i). Let G be a graph with a full degree vertex. Then $\chi_{cd}(G) = \chi(G) = \chi_d(G)$ and $\chi_{cd}(G) = \gamma(G)$ iff $G = K_1$. Also $\chi_{cd}(G) = \chi_\gamma^{cd}(G)$.*
(ii). If $G = K_{m,n}$ then $\chi_{cd}(G) = \chi(G) = \chi_d(G) = \chi_\gamma^{cd}(G) = 2$.
(iii). If G is a bipartite graph with bipartition V_1, V_2 and if V_1 contains a vertex which is adjacent with every vertex of V_2 and V_2 contains a vertex which is adjacent with every vertex of V_1, then $\chi_{cd}(G) = \chi(G) = \chi_\gamma^{cd}(G) = 2$.

Theorem 5.2. *Let G be a bipartite graph with bipartition V_1, V_2 and suppose V_1 contains a vertex which is adjacent with every vertex of V_2. Let $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r}, V_{1,r+1}, \ldots, V_{1,t}\}$, be a minimum partition of V_1 such that $V_{1,i}$, $(1 \leq i \leq t_1)$, is dominated by a vertex of V_2 and each of $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r}\}$*

contains at least two elements and the remaining are singletons. Then $\chi_{cd}(G) = (t+1)$, $\chi_{\gamma}^{cd}(G) = (r+t)$.

Proof. Let $\Pi = \{V_2, V_{1,1}, V_{1,2}, \ldots, V_{1,r}, V_{1,r+1}, \ldots, V_{1,t}\}$. V_2 is dominated by $u \in V_1$. Each $V_{1,i}$, $(1 \leq i \leq r)$ is dominated by a vertex of V_2. Therefore $\chi_{cd}(G) \leq (t+1)$. Suppose $\chi_{cd}(G) = s$. Let

$$\Pi_1 = \{W_1, W_2, \ldots, W_k, W_{k+1}, \ldots, W_s\}$$

be a χ_{cd}-partition of V_2 and $\{W_{k+1}, \ldots, W_s\}$ be a partition of V_1. Let $\Pi_2 = \{V_2, W_{k+1}, \ldots, W_s\}$. Then Π_2 is a *cd*-partition of G and $|\Pi_2| \geq |\Pi_1| = \chi_{cd}(G) = s$. Therefore $k = 1$ and $W_1 = V_2$. Therefore $\Pi_1 = \{V_2, W_{k+1}, \ldots, W_s\}$. Therefore $s - k$ is the least cardinality of a partition of V_1 such that each elements of the partition is dominated by a vertex of V_2. Therefore $s - k = t$. Therefore $\chi_{cd}(G) = s - k + 1 = t + 1$.

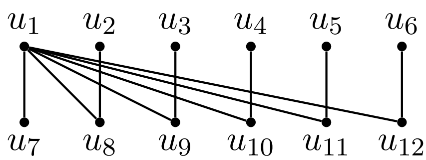
Let $y_1, y_2, \ldots, y_r \in V_2$ dominate $V_{1,1}, V_{1,2}, \ldots, V_{1,r}$ respectively. Let $\Pi_1 = \{V_2 - \{y_2, y_3, \ldots, y_r\}, \{y_2\}, \{y_3\}, \ldots, \{y_r\}, V_{1,1}, V_{1,2}, \ldots, V_{1,r}, V_{1,r+1}, \ldots, V_{1,t}\}$. Then Π_1 is a Π_{γ}^{cd}-partition of G. Therefore $\chi_{\gamma}^{cd}(G) \leq (r+t)$. Let

$$\Pi_2 = \{W_{1,1}, W_{1,2}, \ldots, W_{1,s_1}, W_{2,2}, \ldots, W_{2,s_2}\}$$

be a Π_{γ}^{cd}-partition of G such that $W_{1,1}$, $W_{1,2}$, …, W_{1,s_1} is a minimum partition of V_1, such that each $W_{1,i}$ is dominated by a vertex of V_2 and $\{W_{2,2}, \ldots, W_{2,s_2}\}$ is a partition of V_2. Since $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r}, V_{1,r+1}, \ldots, V_{1,t}\}$, be a minimum partition of V_1 such that $V_{1,i}$, $(1 \leq i \leq t_1)$, is dominated by a vertex of V_2, we get $s_1 = t$. Since s_1 vertices from V_2 are required to dominate V_1, $s_2 \leq s_1$. Therefore $\chi_{\gamma}^{cd}(G) = s_1 + s_2 = t + s_2 \geq t + s_1 = t + t = 2t \geq t + r$. Therefore $\chi_{\gamma}^{cd}(G) = t + r$. □

Remark 5.3. *Since a vertex in* V_1 *dominates every vertex of* V_2*, there exists at least one element of partition of* V_1 *which has at least two elements. Therefore* $r \geq 1$. *Therefore* $\chi_{\gamma}^{cd}(G) = t + r \geq t + 1 = \chi_{cd}(G)$.

Illustration 5.4. *Let* G *be the graph given below:*



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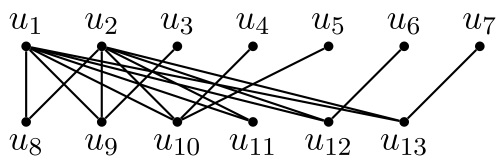
Let $\Pi = \{\{u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\}, \{u_1, u_2\}, \{u_3\}, \{u_4\}, \{u_5\}, \{u_6\}\}$. $\chi_{cd}(G) = 6$. *Let* $\Pi_1 = \{\{u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\}, \{u_1, u_2\}, \{u_3\}, \{u_4\}, \{u_5\}, \{u_6\}\}$. Π_1 *is a* Π_γ^{cd}*-partition of* G*. Also here* $r = 1$, $t = 5$*. Therefore* $\chi_\gamma^{cd}(G) = 6 = t + 1 = \chi_{cd}(G)$.

Remark 5.5. *Let* G *be a bipartite graph. Then* $\chi_\gamma^{cd}(G) = \chi_{cd}(G)$ *iff there exists exactly one subset of* V_1 *with cardinality greater than or equal to two which is dominated by a vertex of* V_2.

Illustration 5.6. *Let* G *be the graph given below:*



G

Let $\Pi = \{\{u_7\}, \{u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}\}, \{u_1, u_2, u_3\}, \{u_4, u_5\}, \{u_6\}\}$ *is a* χ_{cd}*-partition of* G*. Let*

$$\Pi_1 = \{\{u_7\}, \{u_8, u_9, u_{11}, u_{12}, u_{13}\}, \{u_1, u_2, u_3\}, \{u_4, u_5\}, \{u_6\}, \{u_{10}\}\}$$

is a $\Pi_\gamma cd$*-partition of* G. $|\Pi_1| = 4 + 2 = 6 = t + r$ *[where* t *is the minimum number of elements in the partition of* V_1 *such that each element in the partition is dominated by a vertex of* V_2*.* r *is the number of elements in the above partition of* V_1 *which has two or more elements]. Here* $r = 2$, $t = 4$, $\chi_{cd}(G) = 5 = t + 1$ *and* $\chi_\gamma^{cd}(G) = 6 = r + t$.

Remark 5.7. *Let* G *be a bipartite graph with bipartition* V_1, V_2*. Let* $u \in V_1$ *be adjacent with every vertex of* V_2*. Then* $\chi_\gamma^{cd}(G) = \gamma(G) + r - 1$ *where* r *is the number of elements with cardinality greater than or equal to two in a minimum partition of* V_1 *such that each element of the partition is dominated by a vertex of* V_2.

Proof. From above theorem $\chi_{cd}(G) = t + 1 = \gamma(G)$. Also $\chi_\gamma^{cd}(G) = t + r = \gamma(G) + r - 1$. ☐

Remark 5.8. *Let* G *be a bipartite graph with bipartition* V_1, V_2*. Let* $u \in V_1$ *be adjacent with every vertex of* V_2*. Let* r *be the number of elements with cardinality greater than or equal to two in a minimum partition of* V_1 *such that each element of the partition is dominated by a vertex of* V_2*. Then*

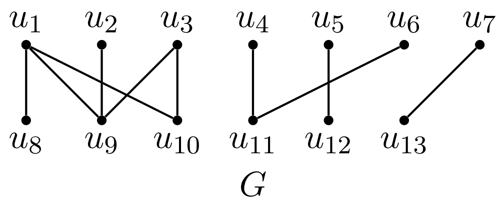
$\chi_\gamma^{cd}(G) = \gamma(G)$ *iff* $r = 1$ *We know from the above remark,* $\chi_\gamma^{cd}(G) = \gamma(G)+r-1$. *Therefore* $\chi_\gamma^{cd}(G) = \gamma(G)$ *iff* $r = 1$.

Remark 5.9. $\chi_\gamma^{cd}(G) = \gamma(G) + r - 1$ *if any one of the* V_1, V_2 *contain a vertex which is adjacent with every vertex of the other partite set. If both* V_1 *and* V_2 *contain* n *vertices which are adjacent with every vertex of the other partite set, then* $\chi_\gamma^{cd}(G) = \gamma(G) = 2$.

Theorem 5.10. *Let* G *be a bipartite graph without isolates with bipartition* V_1, V_2. *Let* $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}, V_{1,r_1+1}, \ldots, V_{1,t_1}\}$ *be a minimum partition of* V_1 *such that each of* $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}\}$ *has cardinality* ≥ 2 *and dominated by* $y_1, y_2, \ldots, y_{r_1} \in V_2$ *respectively with* r_1 *minimum and let* $V_{1,r_1+1}, \ldots, V_{1,t_1}$ *be singletons. Let* $\{V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}, V_{2,r_2+1}, \ldots, V_{2,t_2}\}$ *be a minimum partition of* V_2 *such that each of* $\{V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}\}$ *has cardinality* ≥ 2 *and dominated by* $x_1, x_2, \ldots, x_{r_2} \in V_1$ *respectively with* r_2 *minimum. Then* $\chi_{cd}(G) = t_1 + t_2$ *and* $\chi_\gamma^{cd}(G) = t_1 + t_2 + r_1 + r_2 - s_1 - s_2 - s_3 - s_4$ *where* s_1 *is the number of singletons from* $\{\{y_1\}, \{y_2\}, \ldots, \{y_{r_1}\}\}$ *which coincides with* $V_{2,r_2+1}, \ldots, V_{2,t_2}$, s_2 *is the number of singletons from* $\{\{x_1\}, \{x_2\}, \ldots, \{x_{r_2}\}\}$ *which coincides with* $V_{1,r_1+1}, \ldots, V_{1,t_1}$, s_3 *is the number of singletons from* $\{\{y_1\}, \{y_2\}, \ldots, \{y_{r_1}\}\}$ *which appears one each in* $\{V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}\}$ *and* s_4 *is the number of singletons from* $\{\{x_1\}, \{x_2\}, \ldots, \{x_{r_2}\}\}$ *which appears one each in* $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}\}$. *Therefore* $\chi_\gamma^{cd}(G) = \chi_{cd}(G)$ *iff* $r_1+r_2 = s_1+s_2+s_3+s_4$.

Proof. Let G be a bipartite graph without isolates with bipartition V_1, V_2. Let $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}, V_{1,r_1+1}, \ldots, V_{1,t_1}\}$ be a minimum partition of V_1 such that each of $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}\}$ contains at least two elements and others are singletons and each $V_{1,i}$, $(1 \leq i \leq r_1)$, is dominated by a vertex of V_2. Let $\{V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}, V_{2,r_2+1}, \ldots, V_{2,t_2}\}$ be a minimum partition of V_2 such that each of $V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}$ contains at least two elements and others are singletons and each $V_{2,i}$, $(1 \leq i \leq r_2)$, is dominated by a vertex of $V_{1,i}$. (The partitions are so chosen that r_1, r_2 are minimum.) Let $y_i \in V_2$, $(1 \leq i \leq r_1)$, dominate $V_{1,i}$ and let $x_i \in V_1$, $(1 \leq i \leq r_2)$, dominate $V_{2,i}$. Suppose s_1 of the singletons $\{\{y_1\}, \{y_2\}, \ldots, \{y_{r_1}\}\}$ coincides with $V_{2,r_2+1}, \ldots, V_{2,t_2}$ and let s_2 of the singletons $\{\{x_1\}, \{x_2\}, \ldots, \{x_{r_2}\}\}$ coincides with $V_{1,r_1+1}, \ldots, V_{1,t_1}$. Let s_3 of the singletons $\{\{y_1\}, \{y_2\}, \ldots, \{y_{r_1}\}\}$ appear one each in $\{V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}\}$ and let s_4 of the singletons $\{\{x_1\}, \{x_2\}, \ldots, \{x_{r_2}\}\}$ appears one each in $\{V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}\}$. Let $\Pi = \{\{y_{i,1}\}, \{y_{i,2}\}, \ldots, \{y_{i,r_1-s_1-s_3}\}, \{x_{j,1}\}, \{x_{j,2}\}, \ldots, \{x_{j,r_2-s_2-s_4}\}, V_{1,1}, V_{1,2}, \ldots, V_{1,r_1}, V_{1,r_1+1}, \ldots, V_{1,t_1}, V_{2,1}, V_{2,2}, \ldots, V_{2,r_2}, V_{2,r_2+1}, \ldots, V_{2,t_2}\}$. Then Π is a Π_γ^{cd}-partition of G.Therefore $\Pi_\gamma^{cd}(G) \leq |\Pi| = r_1 - s_1 - s_3 + t_1 + t_2 + r_2 - s_2 - s_4$. By the construction of Π, Π is a Π_γ^{cd}-partition of G of minimum cardinality. Therefore $\chi_\gamma^{cd}(G) = t_1 + t_2 + r_1 + r_2 - s_1 - s_2 - s_3 - s_4$. □

Illustration 5.11. *Let G be the graph given below:*



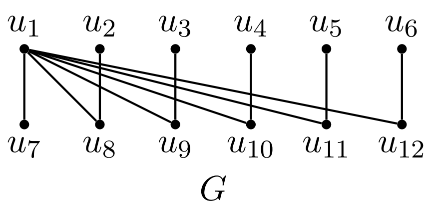
Let $\Pi = \{\{u_1, u_2, u_3\}, \{u_4, u_6\}, \{u_5\}, \{u_7\}, \{u_8, u_9, u_{10}\}, \{u_{11}\}, \{u_{12}\}, \{u_{13}\}\}$. *Here*

$$t_1 = 4, t_2 = 4, r_1 = 2, r_2 = 1, s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 1,$$

$\chi_\gamma^{cd}(G) = t_1 + t_2 + r_1 + r_2 - s_1 - s_2 - s_3 - s_4 = 4 + 4 + 2 + 1 - 1 - 0 - 1 - 1 = 8$. $\{u_1, u_9, u_{11}, u_{12}, u_{13}\}$ *is a minimum dominating set of G. Therefore* $\gamma(G) = 5$. *Note that* $\chi_\gamma^{cd}(G) \neq \gamma(G)$. *Also* $\chi_{cd}(G) = 8$.

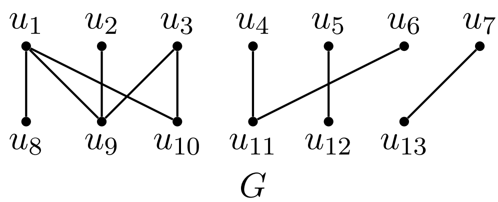
Remark 5.12. $\chi_\gamma^{cd}(G) = \gamma(G)$ *iff* $t_1 = 1$ *or* $t_2 = 1$, $r_1 = 1$, $r_2 = 1$, $s_1 = 0$, $s_2 = 0$, $s_3 = 1$, $s_4 = 1$.

Illustration 5.13. *Let G be the graph given below:*



Let $\Pi = \{\{u_1, u_2\}, \{u_3\}, \{u_4\}, \{u_6\}, \{u_5\}, \{u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\}\}$. $t_1 = 5, t_2 = 1, r_1 = 1, r_2 = 1, s_1 = 0, s_2 = 0, s_3 = 1, s_4 = 1$. $\chi_\gamma^{cd}(G) = t_1 + t_2 + r_1 + r_2 - s_1 - s_2 - s_3 - s_4 = 5 + 1 + 1 + 1 - 0 - 0 - 1 - 1 = 6$. $\{u_1, u_3, u_4, u_5, u_6, u_8\}$ *is a minimum dominating set of G.Therefore* $\gamma(G) = 6$. *Note that* $\chi_\gamma^{cd}(G) = \gamma(G)$.

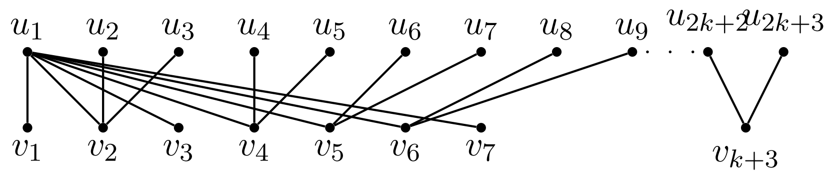
Illustration 5.14. *Let G be the graph given below:*



Let $\Pi = \{\{u_1, u_2, u_3\}, \{u_4, u_6\}, \{u_5\}, \{u_7\}, \{u_8, u_9, u_{10}\}, \{u_{11}\}, \{u_{12}\}, \{u_{13}\}\}$. *Then* Π *is a* χ_{cd}*-partition as well as* χ_γ^{cd}*- partition of* G*.Therefore* $\chi_{cd} = \chi_\gamma^{cd} = 8$. *Here* $t_1 = 4, t_2 = 4, r_1 = 2, r_2 = 1, s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 1$. *Therefore* $r_1 + r_2 = s_1 + s_2 + s_3 + s_4$. *This illustration satisfies the condition of* $\chi_{cd} = \chi_\gamma^{cd}$ *of* G.

Remark 5.15. *Given a positive integer* k*, there exist a connected graph* G *such that* $\chi_\gamma^{cd}(G) - \gamma G = k$.

Proof. Let G be the graph given below:



G

$S = \{u_1, v_2, v_4, v_5, \dots, v_{k+3}\}$ is a minimum dominating set of G. Therefore $\gamma(G) = k + 2$,

$$\chi_\gamma^{cd}(G) = \{\{u_1, u_2, u_3\}, \{u_4, u_5\}, \{u_6, u_7\}, \dots, \{u_{2k+2}, u_{2k+3}\}, \{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}, \dots, \{v_{2k+3}\}\},$$

Π_γ^{cd} is a χ_γ^{cd}-partition of G.Therefore $\chi_\gamma^{cd}(G) = 2k + 2$. Therefore $\chi_\gamma^{cd}(G) - \gamma(G) = k$. □

6. Conclusion

Gamma coloring is a new concept introduced by Sahul Hamid and Gnana Prakasam. Their study was motivated by the presence of a colorful dominating set in any minimum dominator color partition. Color class domination partition is another area where a colorful dominating set can be associated with such a partition. In this paper a study of this concept is initiated. Mainly bipartite graphs are considered for the minimum cardinality of a colorful dominating set in a color class partition. Study of colorful dominating sets with respect to cd-partition in the case of different products of two graphs can be made. The relation between $\chi_\gamma^{cd}(G)$ and $\chi_\gamma^{cd}(\mu(G))$, where $\mu(G)$ is the Mycielskian of

G, can be derived. Characterization of graphs for which $\chi_d(G) = \chi_\gamma^{cd}(G)$ or $\chi_\gamma^{cd}(G) = \chi_{cd}(G)$ may be found.

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