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NEW NUMERICAL METHOD FOR SOLVING DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we introduce a new numerical method for solving ordinary differential equations in both linear and non linear cases. We apply new iterative method (NIM) on implicit midpoint rule to derive a new numerical method. We further present the error, convergence and stability analysis of the proposed method. The efficiency of the new method is tested through various types of numerical problems and it shows that the results are be the same as with exact solutions.

AMS Subject Classification: 65L04, 65L05, 34G20, 65L20

Key Words: new iterative method (NIM), ordinary differential equations, implicit midpoint formula, error analysis, stability, convergence

1. Introduction

Differential equations play a prominent role in the study of Physical, Chemical, Engineering, Economics and Biological fields. Also, it observed that most of natural phenomena leads to non linear differential equations. For such differential equations it difficult to find the solution by using analytical methods. In such cases the numerical methods plays a crucial role for obtain-

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ing the solution. Nowadays the construction of stable, time efficient and accurate numerical methods for the solution of differential equations has been considered widely. Recently, the method developed by Daftardar-Gejji and Jafari(NIM)[5] is powerful technique for solving wide range of non linear equations [8, 11, 17, 16, 14, 1, 2, 3, 6] and Patade and Bhalekar [13] proposed new method by applying Daftardar-Gejji and Jafari technique on the implicit trapezium method to get new second order formula. In this paper, we have employ the power of NIM to construct a new numerical method for solving ordinary differential equations and discuss error, stability and convergence of proposed method with suitable examples using software package maple. We have organised the paper as follows.

In section 2, we discussed the preliminary concepts. The NIM is described briefly in section 3. A new numerical method using NIM is presented in section 4. Analysis of this numerical method is given in section 5. Test problem is shows that efficiency of the proposed method in section 6 and finally conclusions are summarized.

2. Preliminary

In this section, we discuss some basic definitions and results [4, 15, 12, 9]. Consider the initial value problem

$$y^{'} = f(x, y), \qquad y^i(x_0) = \eta, \qquad i = 1, 2, 3, ..., n \tag{2.1}$$

where $y : [a, b] \to \mathbb{R}^n, \eta \in \mathbb{R}^n$ and $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

Definition 1. Single step method

A general single step method can be written in the form

$$y_{n+1} = y_n + h\Phi(x_n, y_n, h), \qquad n = 0, 1, 2, ... \tag{2.2}$$

where Φ is called the increment function.

Definition 2. Local truncation error

The local truncation error or discretization error of the numerical method is the difference between the true (exact) solution $y(x_{n+1})$ and the numerical solution determined from numerical method y_{n+1}. Therefore, the local truncation error is given by

$$T_{n+1} = y(x_{n+1}) - y_{n+1} \tag{2.3}$$

Definition 3. Order of a method

The order of the single step method is the largest integer p for which

$$\left|\frac{1}{h}T_{n+1}\right| = O(h^p) \tag{2.4}$$

Definition 4. Consistency of single step method
A single step method (2.2) is called consistent if

$$\Phi(x, y, 0) = f(x, y) \tag{2.5}$$

Definition 5. A single step method (2.2) is said to be regular if the function $\Phi(x, y, h)$ is defined and continuous in the domain $a \le x \le b, -\infty < y^i < \infty, i = 1, 2, ..., n, 0 \le h \le h_0$ and if there exist a constant L such that

$$\|\Phi(x, y, h) - \Phi(x, y^*, h)\| \le L\|y - y^*\|$$

for every $x \in [x_0, b]$, $\quad y, y^* \in (-\infty, \infty)$, $\quad h \in (0, h_0)$.

Theorem 6. *Suppose the single step method* (2.2) *is regular. Then the relation* (2.5) *is a necessary and sufficient condition for the convergence of the method defined by* (2.2).

3. New Iterative Method(NIM)

In this section, we describe a new iterative method introduced by Daftardar-Gejji and Jafari(NIM) [5] for solving non linear functional equation of the form

$$u = f + N(u) \tag{3.1}$$

where f is a known function and N is a non linear operator. It is assumed that NIM provides the solution to the equation (3.1) in the form of series

$$u = \sum_{i=0}^{\infty} u_i \tag{3.2}$$

In this method non linear operator N is decomposed by NIM as bellow:

$$\begin{aligned} N(u) = N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \\ &= \sum_{i=0}^{\infty} G_i \end{aligned} \tag{3.3}$$

where $G_0 = N(u_0)$ and $G_i = \Big\{N\Big(\sum_{j=0}^{i} u_j\Big) - N\Big(\sum_{j=0}^{i-1} u_j\Big)\Big\}, \quad i \geq 1.$

From equations (3.2) and (3.3) , equation (3.1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} G_i \tag{3.4}$$

Now we define the recurrence relation from the equation (3.4) as bellow:

$$\begin{aligned} u_0 &= f \\ u_1 &= N(u_0) \\ u_{m+1} &= N(u_0 + u_1 + ... + u_m) - N(u_0 + u_1 + ... + u_{m-1}), \quad m = 1, 2, ... \end{aligned}$$

Thus the K-term approximate solution is given by

$$u = \sum_{i=0}^{k-1} u_i \tag{3.5}$$

for suitable integer k.

4. New Numerical Method

In this section, we developed a new numerical method based on NIM.

Now, let us consider y_{n+1} ia an approximate solution to $y(x_{n+1})$, then implicit midpoint rule [10] is given by

$$y_{n+1} = y_n + hf\Big(x_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1})\Big) \tag{4.1}$$

where $h = x_{n+1} - x_n$ and $x_n = x_0 + nh, \quad n = 0, 1, 2, ...$

We can rewrite equation (4.1) as the form of equation (3.1) by consider

$$\begin{aligned} u &= y_{n+1} \\ f &= y_n \\ N(u) &= hf\Big(x_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1})\Big) \end{aligned}$$

Now, let us apply NIM on equation (4.1) to get 4-term solution as

$$\begin{aligned} u &= u_0 + u_1 + u_2 + u_3 \\ &= u_0 + N(u_0) + N(u_0 + u_1) - N(u_0) + N(u_0 + u_1 + u_2) - N(u_0 + u_1) \\ &= u_0 + N(u_0 + N(u_0) + N(u_0 + u_1) - N(u_0)) \\ &= u_0 + N(u_0 + N(u_0 + N(u_0))) \end{aligned}$$

which is,

$$y_{n+1} = y_n + N\Big(y_n + N\Big(y_n + hf\Big(x_n + \frac{h}{2}, y_n\Big)\Big)\Big)$$

or

$$y_{n+1} = y_n + hf\Big(x_n + \frac{h}{2}, y_n + \frac{h}{2}f\Big(x_n + \frac{h}{2}, y_n + \frac{h}{2}f\Big(x_n + \frac{h}{2}, y_n\Big)\Big)\Big) \tag{4.2}$$

If we set

$$K_1 = f\Big(x_n + \frac{h}{2}, y_n\Big) \tag{4.3}$$

$$K_2 = f\Big(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1\Big) \tag{4.4}$$

$$K_3 = f\Big(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2\Big) \tag{4.5}$$

Now equation (4.2) becomes

$$y_{n+1} = y_n + hK_3 \tag{4.6}$$

4.1. Non Runge-Kutta Method

In the following discussion we proved that the new numerical method is not a Runge-Kutta method. We write (4.2) in Runge-Kutta form as

$$\begin{aligned} y_{n+1} &= y_n + h(b_1K_1 + b_2K_2 + b_3K_3) \\ K_i &= f(x_n + c_ih, y_n + h(a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)), \quad i = 1, 2, 3. \end{aligned}$$

Note that $b_1 = b_2 = 0, b_3 = 1$; $c_1 = c_2 = c_3 = \frac{1}{2}$; $a_{11} = a_{12} = a_{13} = 0, a_{21} = \frac{1}{2}, a_{22} = a_{23} = 0, a_{31} = 0, a_{32} = \frac{1}{2}, a_{33} = 0$.

For this method the Butcher tableau is as follows:

$$\begin{array}{c|ccc} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \hline & 0 & 0 & 1 \end{array}$$

For any Runge-Kutta method [4] it is necessary that the terms satisfy the order condition $\sum_{i=1}^{3} a_{in} = c_i, \quad \forall i$. From the above Butcher tableau, $a_{11} + a_{12} + a_{13} = 0 \neq c_1$. This proves that the proposed method is different from Runge-Kutta method.

5. Analysis of New Numerical Method

In the present section we discussed the error, convergence and stability analysis of new numerical method respectively.

5.1. Error Analysis

Theorem 7. *The new numerical method defined by* (4.6) *is of second order.*

Proof: The Taylor's series expansion of $y(x_{n+1})$ may be written as

$$\begin{aligned} y(x_{n+1}) =& y_n + hf_n + \frac{h^2}{2} f_{n,x} + \frac{h^2}{2} f_n f_{n,y} + \frac{h^3}{6} f_{n,xx} + \frac{h^3}{3} f_n f_{n,xy} + \frac{h^3}{6} f_{n,x} f_{n,y} \\ &+ \frac{h^3}{6} f_n f_{n,y}^2 + \frac{h^3}{6} f_n^2 f_{n,yy} + O(h^4), \end{aligned} \tag{5.1}$$

where $f_n = f(x_n, y_n)$, $f_{n,x} = \left(\frac{\partial f(x,y)}{\partial x}\right)_{(x_n,y_n)}$, $f_{n,y} = \left(\frac{\partial f(x,y)}{\partial y}\right)_{(x_n,y_n)}$, etc.

Using second order Taylor's series we can write K_1, K_2 and K_3 as

$$K_1 = f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}, \tag{5.2}$$

$$\begin{aligned} K_2 =& f_n + \frac{h}{2} f_{n,x} + \frac{h}{2}\left(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\right) f_{n,y} + \frac{h^2}{8} f_{n,xx} \\ &+ \frac{h^2}{4}\left(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\right) f_{n,xy} + \frac{h^2}{8}\left(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\right)^2 f_{n,yy}, \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
K_3 =& f_n + \frac{h}{2} f_{n,x} + \frac{h}{2}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h}{2}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big) f_{n,y} + \frac{h^2}{8} f_{n,xx} \\
&+ \frac{h^2}{4}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big) f_{n,xy} + \frac{h^2}{8}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big)^2 f_{n,yy}\Big) f_{n,y} \\
&+ \frac{h^2}{8} f_{n,xx} + \frac{h^2}{4}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h}{2}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big) f_{n,y} + \frac{h^2}{8} f_{n,xx} \\
&+ \frac{h^2}{4}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big) f_{n,xy} + \frac{h^2}{8}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big)^2 f_{n,yy}\Big) f_{n,xy} \\
&+ \frac{h^2}{8}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h}{2}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big) f_{n,y} + \frac{h^2}{8} f_{n,xx} \\
&+ \frac{h^2}{4}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big) f_{n,xy} + \frac{h^2}{8}\Big(f_n + \frac{h}{2} f_{n,x} + \frac{h^2}{8} f_{n,xx}\Big)^2 f_{n,yy}\Big)^2 f_{n,yy}.
\end{aligned} \tag{5.4}
$$

Using equation (5.4) in equation (4.6), we obtain

$$
\begin{aligned}
y_{n+1} =& y_n + h f_n + \frac{h^2}{2} f_{n,x} + \frac{h^2}{2} f_n f_{n,y} + \frac{h^3}{4} f_{n,x} f_{n,y} + \frac{h^3}{4} f_n f_{n,y}^2 + \frac{h^3}{8} f_{n,xx} \\
&+ \frac{h^3}{4} f_n f_{n,xy} + \frac{h^3}{8} f_n^2 f_{n,yy} + O(h^4).
\end{aligned} \tag{5.5}
$$

By equation (2.3), we have

$$
\begin{aligned}
T_{n+1} =& y(x_{n+1}) - y_{n+1} \\
T_{n+1} =& \frac{h^3}{6} f_{n,xx} + \frac{h^3}{3} f_n f_{n,xy} + \frac{h^3}{6} f_{n,x} f_{n,y} + \frac{h^3}{6} f_n f_{n,y}^2 + \frac{h^3}{6} f_n^2 f_{n,yy} - \frac{h^3}{4} f_{n,x} f_{n,y} \\
&- \frac{h^3}{4} f_n f_{n,y}^2 - \frac{h^3}{8} f_{n,xx} - \frac{h^3}{4} f_n f_{n,xy} - \frac{h^3}{8} f_n^2 f_{n,yy} + O(h^4), \\
T_{n+1} =& h^3\Big(\frac{f_{n,xx}}{24} + \frac{f_n f_{n,xy}}{12} - \frac{f_{n,x} f_{n,y}}{12} - \frac{f_n f_{n,y}^2}{12} + \frac{f_n^2 f_{n,yy}}{24}\Big) + O(h^4).
\end{aligned}
$$

Thus using definition (3), the new numerical method (4.6) is of second order.

5.2. Convergence Analysis

Theorem 8. *Consider the function $f(x, y)$ is defined and continuous in the strip $S(|x - x_0| \leq a, \|y\| < \infty, a > 0)$ and satisfy the Lipschitz condition*

$$\|f(x, y) - f(x, y^*)\| \leq L\|y - y^*\|$$

for every $(x, y), (x, y^*) \in S$, where L is Lipschitz constant, then the method (4.6) is convergent.

Proof. Let us assume that the increment function,

$$\Phi(x_n, y_n, h) = \frac{1}{h}(hK_3),$$
$$\Phi(x_n, y_n, h) = K_3. \tag{5.6}$$

From equation (4.3) - (4.5), we have

$$\Phi(x, y, 0) = f(x, y).$$

Thus the method (4.6) by definition (4) is consistent.

Denote $K_1^* = f\big(x_n + \frac{h}{2}, y_n^*\big), K_2^* = f\big(x_n + \frac{h}{2}, y_n^* + \frac{h}{2}K_1^*\big)$ and $K_3^* = f\big(x_n + \frac{h}{2}, y_n^* + \frac{h}{2}K_2^*\big)$ for every $(x, y), (x, y^*) \in S$ and $K_i's (i = 1, 2, 3)$ are defined in (4.3) - (4.5). Since $f(x, y)$ is Lipschitz, we have

$$\begin{aligned}
\|K_1 - K_1^*\| &= \Big\|f\Big(x_n + \frac{h}{2}, y_n\Big) - f\Big(x_n + \frac{h}{2}, y_n^*\Big)\Big\|, \\
&\leq L\|y_n - y_n^*\|. \\
\|K_2 - K_2^*\| &= \Big\|f\Big(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1\Big) - f\Big(x_n + \frac{h}{2}, y_n^* + \frac{h}{2}K_1^*\Big)\Big\|, \\
&\leq L\Big\|y_n + \frac{h}{2}K_1 - y_n^* - \frac{h}{2}K_1^*\Big\|, \\
&\leq L\Big(1 + \frac{hL}{2}\Big)\|y_n - y_n^*\|. \\
\|K_3 - K_3^*\| &= \Big\|f\Big(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2\Big) - f\Big(x_n + \frac{h}{2}, y_n^* + \frac{h}{2}K_2^*\Big)\Big\|, \\
&\leq L\Big\|y_n + \frac{h}{2}K_2 - y_n^* - \frac{h}{2}K_2^*\Big\|, \\
&\leq L\Big(1 + \frac{hL}{2} + \frac{h^2L^2}{4}\Big)\|y_n - y_n^*\|.
\end{aligned}$$

Using equation (5.6), we have

$$\begin{aligned}
\|\Phi(x_n, y_n, h) - \Phi(x_n, y_n^*, h)\| &= \|K_3 - K_3^*\|, \\
&\leq L\Big(1 + \frac{hL}{2} + \frac{h^2L^2}{4}\Big)\|y_n - y_n^*\|, \\
&\leq L^{'}\|y_n - y_n^*\|,
\end{aligned}$$

where $L^{'} = L\Big(1 + \frac{hL}{2} + \frac{h^2L^2}{4}\Big)$.

Hence the increment function Φ satisfies a Lipschitz condition in y and thus by applying definition (5) the new method (4.6) is regular.

Hence from theorem (6), the new numerical method (4.6) is convergent. □

5.3. Stability Analysis

Consider the test equation,

$$y' = \lambda y \tag{5.7}$$

Now applying the new method (4.6) to the test equation (5.7), we get

$$\begin{aligned} K_1 &= \lambda y_n. \\ K_2 &= \lambda\Big(y_n + \frac{h}{2}K_1\Big), \\ &= \lambda\Big(1 + \frac{h\lambda}{2}\Big)y_n. \\ K_3 &= \lambda\Big(y_n + \frac{h}{2}K_2\Big), \\ &= \lambda\Big(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{4}\Big)y_n. \end{aligned}$$

Now using equation (4.6), we can write

$$y_{n+1} = \Big(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{4}\Big)y_n.$$

Hence the stability region of the new numerical method is given by the inequality

$$\Big|1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{4}\Big| \leq 1,$$

where $\lambda = Re(\lambda) + iIm(\lambda)$ is a complex number.

The stability region is shown in figure (1).

6. Numerical Test Problems

We present some problems and show that the solution of new method is good agreement with exact solutions. Maple software is used for this calculations.

Problem 6.1. Consider the nonlinear initial value problem [12]

$$y' = -2xy^2, \quad y(0) = 1 \tag{6.1}$$

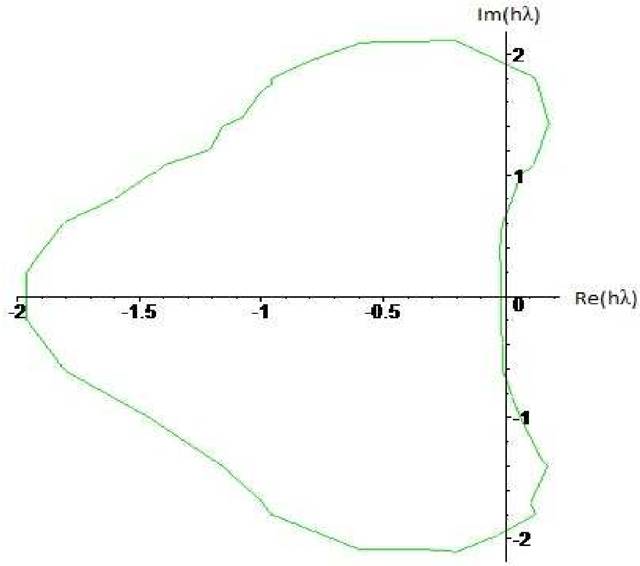

Figure 1: Stability region of new numerical method.

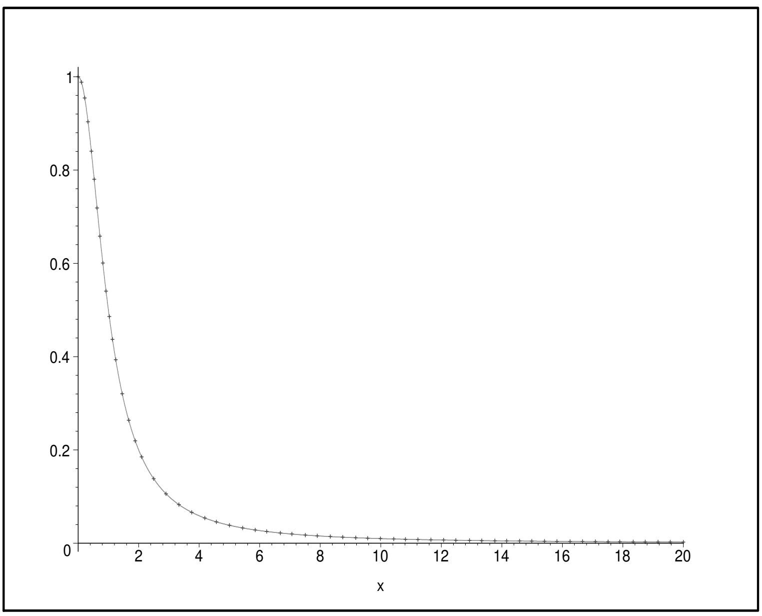

Figure 2: Comparison of solutions of problem (6.1)

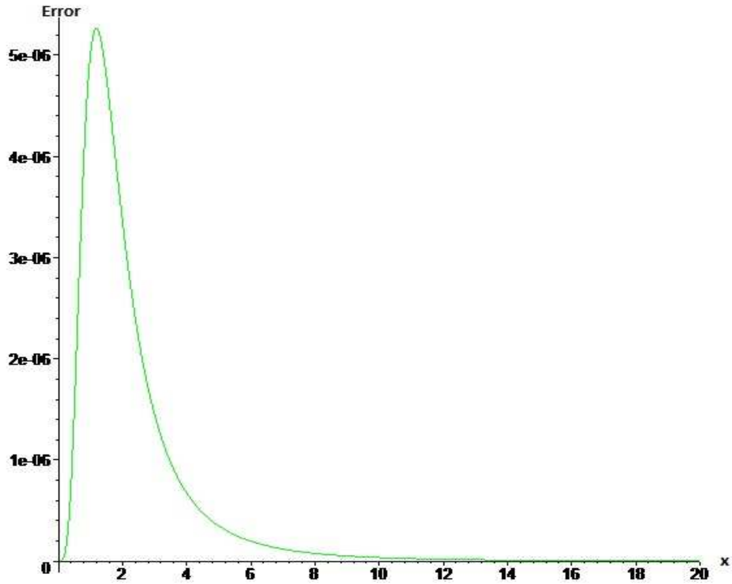

Figure 3: Absolute error in problem (6.1)

The exact solution of the problem (6.1) is

$$y(x) = \frac{1}{1+x^2}$$

We compare the solution of problem (6.1) obtained by new method (line) and exact solution (point) in figure (2). It is observed that numerical solution is good agreement with exact solution. The absolute error in the new method is plotted in figure (3) and maximum absolute error is 5.26617×10^{-6}.

Problem 6.2. Consider the nonlinear initial value problem

$$\frac{dy}{dx} = y^2 - xy^2, \quad y(0) = 1 \tag{6.2}$$

The exact solution of problem (6.2) is

$$y(x) = \frac{2}{x^2 - 2x + 2}$$

In figure (4), we compare the graph of solution by new method (line) and exact solution (point). It is observed that our solution is good agreement with exact solution. The absolute error in the new numerical method is plotted in figure (5) and maximum absolute error is 3.72231×10^{-5}.

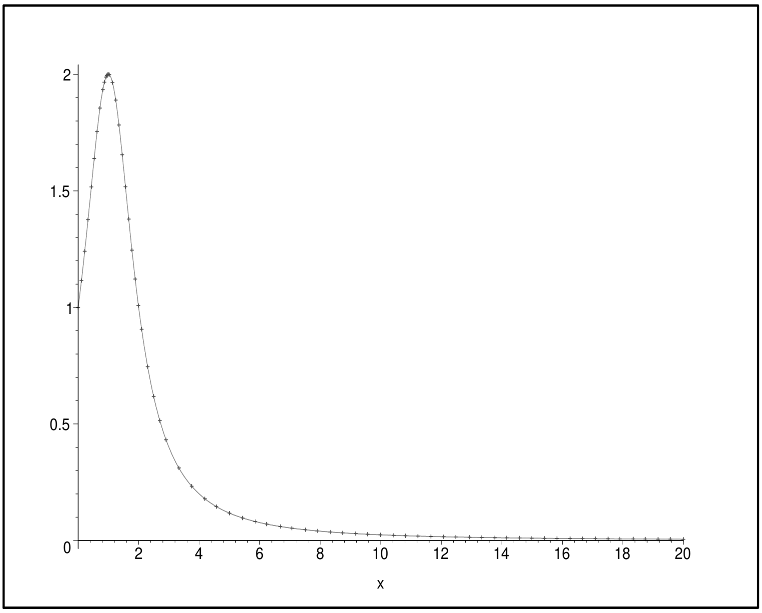

Figure 4: Comparison of solutions of problem (6.2)

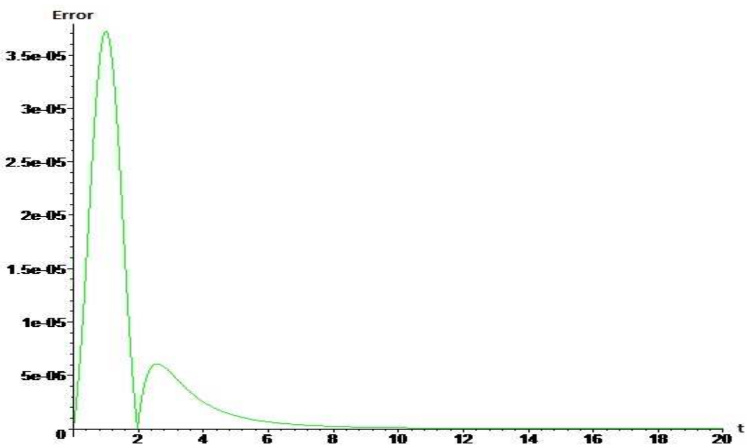

Figure 5: Absolute error in problem (6.2)

Problem 6.3. Consider a stiff system of differential equation [7]

$$\frac{dx}{dt} = -1002x + 1000y^2$$
$$\frac{dy}{dt} = x - y - y^2, \quad t \in [0, 5] \tag{6.3}$$

with initial conditions

$$x(0) = 1, \quad y(0) = 1$$

The exact solution of problem (6.3) is

$$x(t) = e^{-2t} \quad and \quad y(t) = e^{-t}$$

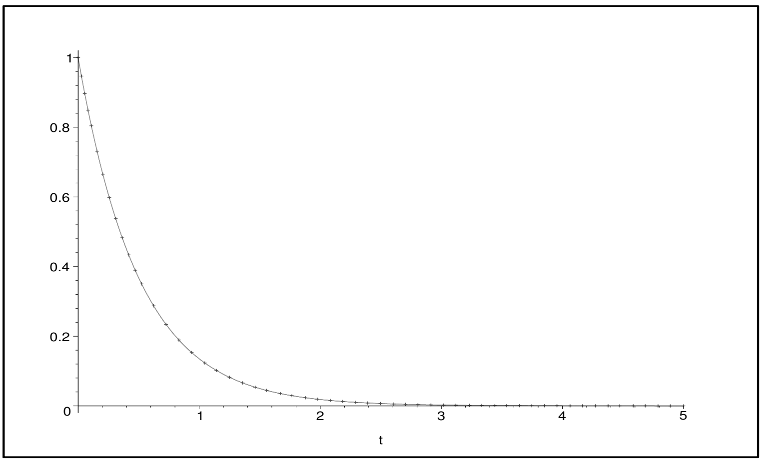

Figure 6: Comparison of solutions $x(t)$
of problem (6.3)

We are compared the numerical solutions (line) and exact solutions (point) (where both the graphs coincide) of $x(t)$ and $y(t)$ in figure (6) and (7) respectively and the absolute errors are plotted in figure (8) and (9).

The maximum absolute error in $x(t)$ and $y(t)$ in the interval $t \in [0, 5]$ is 4.9639×10^{-7} and 3.067×10^{-8} respectively.

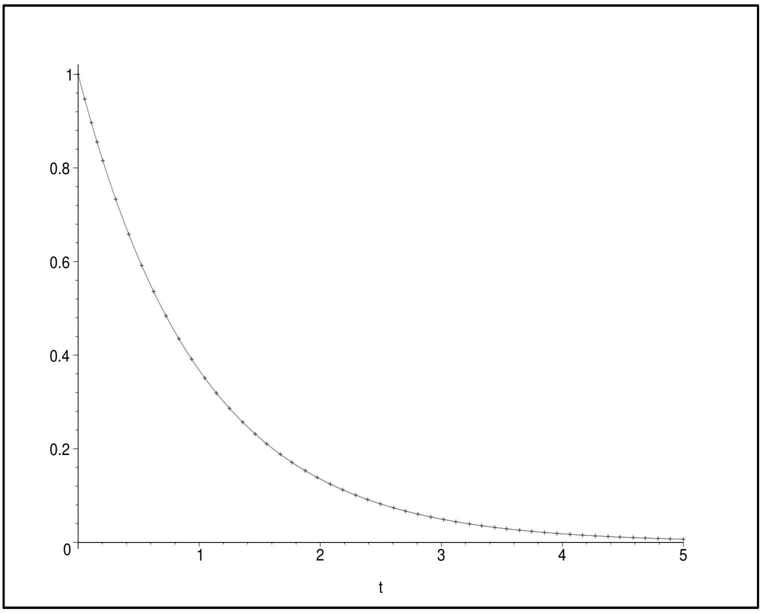

Figure 7: Comparison of solutions $y(t)$ of problem (6.3)

7. Conclusions

In this paper, we have developed a new efficient numerical method which is simple and powerful for solving differential equations for linear as well as non-linear. We have used new iterative method on implicit midpoint rule. Further, we analyzed the order, consistency and the stability for the new method. The numerical test problems have shown that our solutions are in good agreement with exact solutions. It observed that this method can be used effectively to solve stiff systems of differential equations.

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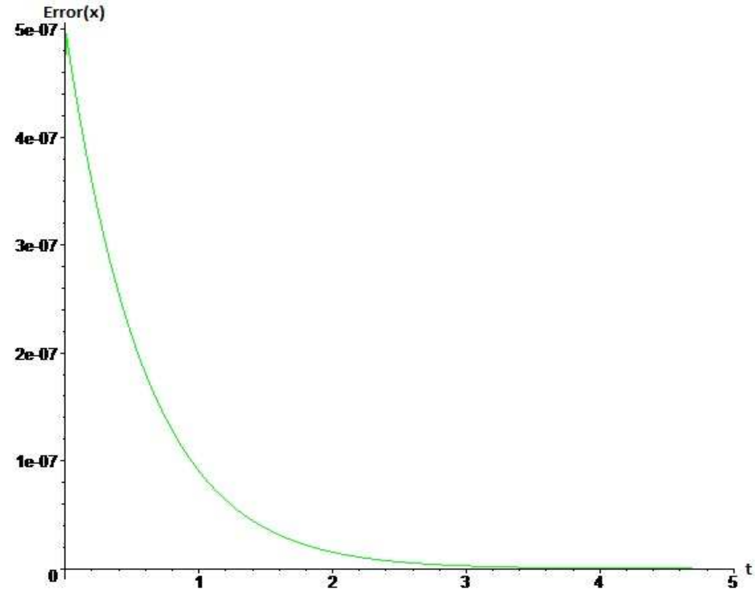

Figure 8: Absolute error in $x(t)$ for problem (6.3)

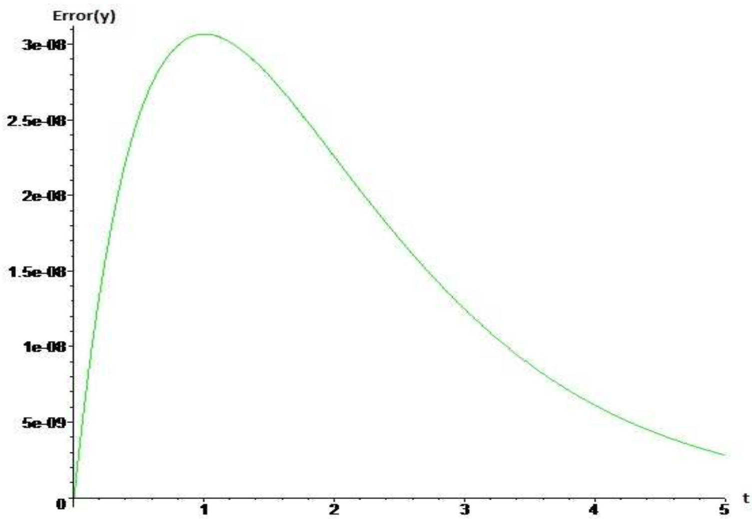

Figure 9: Absolute error in $y(t)$ for problem (6.3)

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Table 1: Comparison of numerical results for problem (6.3)

t	Solutions	Exact solution	Our solution	Absolute error
0	$x(t)$	1	1	0
	$y(t)$	1	1	0
0.5	$x(t)$	0.36787944117	0.36787922602	0.21515×10^{-6}
	$y(t)$	0.60653065971	0.60653063455	0.2516×10^{-7}
1	$x(t)$	0.13533528324	0.13533519276	0.9048×10^{-7}
	$y(t)$	0.36787944117	0.36787941050	0.3067×10^{-7}
1.5	$x(t)$	0.049787068368	0.049787030909	0.37459×10^{-7}
	$y(t)$	0.22313016015	0.22313013219	0.2796×10^{-7}
2	$x(t)$	0.018315638889	0.018315623591	0.15298×10^{-7}
	$y(t)$	0.13533528324	0.13533526067	0.2257×10^{-7}
2.5	$x(t)$	0.0067379469991	0.0067379408060	0.61931×10^{-8}
	$y(t)$	0.082084998624	0.082084981491	0.17133×10^{-7}
3	$x(t)$	0.0024787521767	0.0024787496913	0.24854×10^{-8}
	$y(t)$	0.049787068368	0.049787055896	0.12472×10^{-7}
3.5	$x(t)$	0.00091188196555	0.00091188097544	0.99011×10^{-9}
	$y(t)$	0.030197383422	0.030197374603	0.8819×10^{-8}
4	$x(t)$	0.00033546262790	0.00033546223580	0.39210×10^{-9}
	$y(t)$	0.018315638889	0.018315632779	0.6110×10^{-8}
4.5	$x(t)$	0.00012340980409	0.00012340964943	0.15466×10^{-9}
	$y(t)$	0.011108996538	0.011108992364	0.4174×10^{-8}
5	$x(t)$	0.000045399929762	0.000045399869063	0.60699×10^{-10}
	$y(t)$	0.0067379469991	0.0067379441849	0.28142×10^{-8}

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