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SOME NOTES ON THE KUMARASWAMY–WEIBULL–EXPONENTIAL CUMULATIVE SIGMOID

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Abstract: In this paper we study the one-sided Hausdorff approximation of the shifted Heaviside step function by a family of Kumaraswamy-Weibull-Exponential cumulative sigmoid. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

We also look at a possible extension, which we call γ-*Family of Kumaraswamy-Weibull-Exponential cumulative sigmoid*.

AMS Subject Classification: 68N30, 41A46

Key Words: Kummaraswamy-Weibull-generated family, Kumaraswamy-Weibull-Exponential cumulative sigmoid, γ-Family of Kumaraswamy-Weibull Exponential cumulative sigmoid, Heaviside function, Hausdorff approximation, upper and lower bounds

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1. Introduction

The Weibull distribution has been widely used in survival and reliability analyses.

Some modifications, properties and applications of Weibull and Weibull-R families of distributions can be found in [1]-[9].

The cumulative distribution function (cdf) of Kummaraswamy-Weibull-generated family is given by

$$M^*(t) = 1 - \left(1 - \left(1 - e^{-\alpha\left(\frac{G(t)}{1-G(t)}\right)^{\beta}}\right)^{a}\right)^{b}, \tag{1}$$

where $t > 0$, $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$.

Some of the generated families can be found in [1], [10]-[15]. Fot other results, see [17]-[22].

In [16], the authors proposed a five-parameters Kumaraswamy-Weibull-exponential cumulative sigmoid:

$$M(t) = 1 - \left(1 - \left(1 - e^{-\alpha\left(e^{\lambda t}-1\right)^{\beta}}\right)^{a}\right)^{b}, \tag{2}$$

where $t > 0$, $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$.

Definition 1. The *shifted Heaviside step function* is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

Definition 2. The Hausdorff distance [23] (the H-distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$.

More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In this note we study the Hausdorff approximation of the *shifted Heaviside step function* by the family of type (2).

2. Main Results

We consider the following class of this family:

$$M(t) = 1 - \left(1 - \left(1 - e^{-\alpha\left(e^{\lambda t}-1\right)^{\beta}}\right)^{a}\right)^{b}, \tag{3}$$

with

$$t_0 = \frac{1}{\lambda}\ln\left(1 + \left(-\frac{1}{\alpha}\ln\left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)\right)^{\frac{1}{\beta}}\right); \quad M(t_0) = \frac{1}{2}. \tag{4}$$

The one-sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - ((3)-(4)) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{5}$$

The following theorem gives upper and lower bounds for d

Theorem. Let

$$\begin{aligned} p &= -\tfrac{1}{2}, \\ q &= 1 + ab\alpha\beta\lambda\left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right) \\ &\times\left(1 + \left(-\tfrac{1}{\alpha}\ln\left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)\right)^{\frac{1}{\beta}}\right) \\ &\times\left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{a-1}{a}}\left(-\tfrac{1}{\alpha}\ln\left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)\right)^{\frac{\beta-1}{\beta}} 0.5^{\frac{b-1}{b}} \\ r &= 2.1q. \end{aligned} \tag{6}$$

For the one-sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid ((3)-(4)) the following inequalities hold for $q > \frac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{7}$$

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{8}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{9}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1). In addition $G'(d) > 0$.

Further, for $q > \frac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

3. Numerical examples

The model ((3)-(4)) for $\beta = 2$; $\lambda = 0.25$; $a = 1.1$; $b = 8$; $\alpha = 0.5$, $t_0 = 1.53782$ is visualized on Fig. 2.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.318568$, $d_l = 0.299455$, $d_r = 0.36108$.

The model ((3)-(4)) for $\beta = 1.9$; $\lambda = 0.2$; $a = 0.25$; $b = 7.5$; $\alpha = 1.2$, $t_0 = 0.0273433$ is visualized on Fig. 3.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.155909$, $d_l = 0.0650123$, $d_r = 0.17769$.

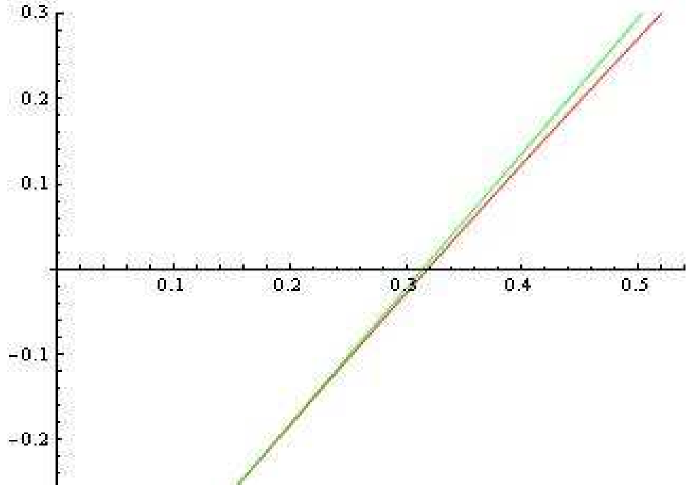

Figure 1: The functions $F(d)$ and $G(d)$ for $\beta = 2$; $\lambda = 0.25$; $a = 1.1$; $b = 8$; $\alpha = 0.5$.

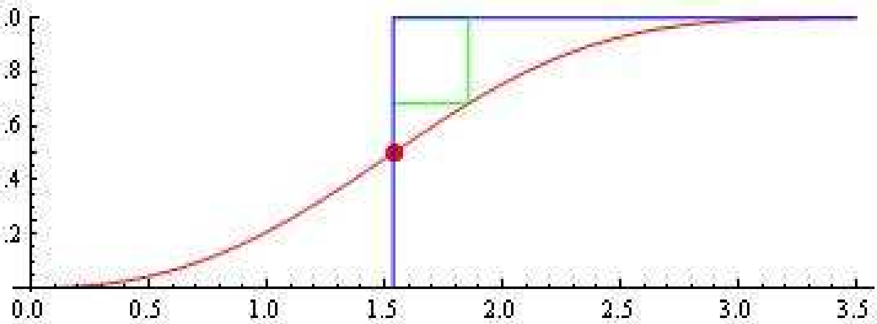

Figure 2: The model ((3)-(4)) for $\beta = 2$; $\lambda = 0.25$; $a = 1.1$; $b = 8$; $\alpha = 0.5$, $t_0 = 1.53782$, $t_0 = 0.435501$; H-distance $d = 0.318568$, $d_l = 0.299455$, $d_r = 0.36108$.

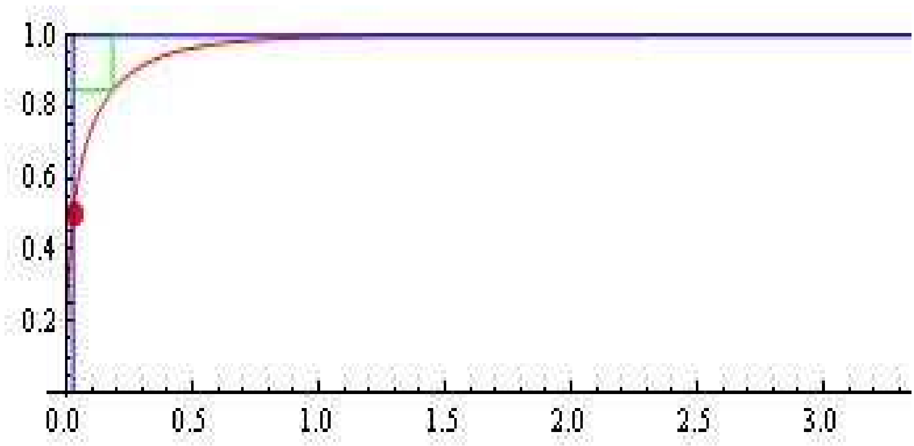

Figure 3: The model ((3)-(4)) for $\beta = 1.9$; $\lambda = 0.2$; $a = 0.25$; $b = 7.5$; $\alpha = 1.2$, $t_0 = 0.0273433$, $t_0 = 0.332504$; H-distance $d = 0.155909$, $d_l = 0.0650123$, $d_r = 0.17769$.

From the above examples, it can be seen that the proven estimates (see Theorem) for the value of the Hausdorff approximation is reliable when assessing the important characteristic - "saturation".

Remark. We define the following γ-*Family of Kumaraswamy-Weibull-Exponential cumulative sigmoid*

$$M_\gamma(t) = \left(1 - \left(1 - \left(1 - e^{-\alpha\left(e^{\lambda t}-1\right)^\beta}\right)^a\right)^b\right)^\gamma, \tag{10}$$

where $t > 0$, $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $\gamma > 0$.

The sigmoid (10) for $\beta = 2$; $\lambda = 0.25$; $a = 1.1$; $b = 8$; $\alpha = 0.5$ and $\gamma = 2.1$ is visualized on Fig. 4.

Based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems for the general model $M_\gamma(t)$ on his/her own.

The proposed new model can be successfully used to approximating data from Population Dynamic, Biostatistics and Debugging Theory.

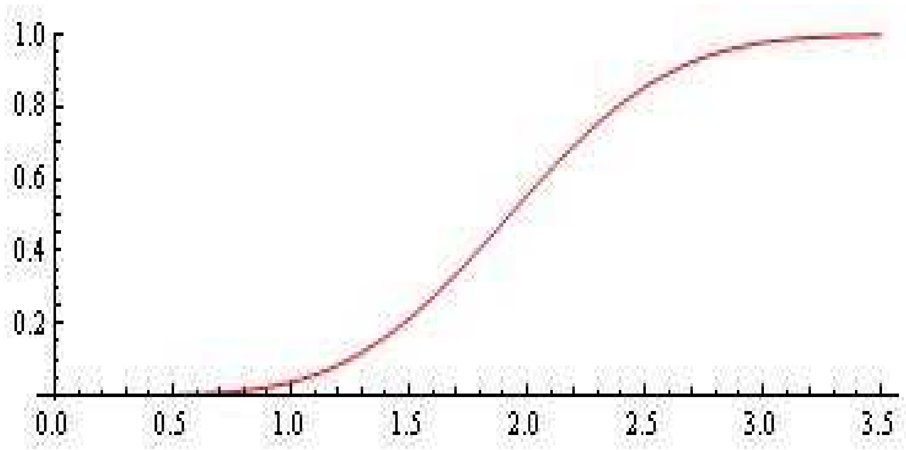

Figure 4: The model (10) for $\gamma = 2.1$; $\beta = 1.9$; $\lambda = 0.2$; $a = 0.25$; $b = 7.5$; $\alpha = 1.2$.

For some approximation, computational and modelling aspects, see [24]-[37].

Some software reliability models, can be found in [38]-[40].

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