

A NUMERICAL INVESTIGATION OF AN INVERSE PROBLEM IN THERMODYNAMICS

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Abstract: The problem of finding the coefficients and parameters for thermodynamical problems when the solution is known is investigated. Our model problem consists of a sample, inside a cylinder, that is completely insulated except at front where the heat source is located and at the rear end $x = L$ where it is open to a cooler support rod with limited insulation. The direct and inverse problems are described and numerical results are given for the parameters of the model problem.

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1. Introduction

The problem of finding the coefficients and parameters for thermodynamical problems when the solution is known is an inverse problem. To solve the inverse

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problem, one must first solve the direct problem, then solve the inverse problem for some coefficients and parameters. Solving such a problem therefore requires solving an optimization (minimization) problem, which is algorithmically more challenging than the linear problem. These problems have many applications in engineering and science. To cite few in [6] F. Yaman et al. provided a wide range review paper on inverse problems which focus on applied sciences. In [2] Z.-L. Deng et al. investigated an inverse heat conduction problem on a semi-infinite domain in which the fractional diffusion equation is solved analytically in the direct problem and a mollification method with Dirichlet kernel is adopted for the solution of the inverse problem. In the context of linear inverse problems, this point has been raised by Backus and Gilbert (see [1]) and more recently by Parker (see [3]). An Example of an inverse thermodynamics problem related to the relative entropy is described in [5]. The book by A. Samarski and P. N. Vabishchevich [4] gives a good overview of many computational aspects of the subject.

In the inverse problem, the goal is to reconstruct the model problem from a set of measurements.

2. The model problem

Our thermodynamic model problem consists on of sample that is completely insulated except at $x = 0$ where the heat source is located. The right hand side, the sample end at $x = L$, is not truly insulated but open to a cooler support rod with limited insulation. We assume the sample is initially at the uniform temperature, U , heated by radiation from the pipe, which is in a horizontal position being inductively heated at a constant known temperature. From the outside view one can see a long pipe held in place by graphite wool insulation. Right in the center is a another small solid, called the sample (see Fig. 1). A cross section of the model is shown in Fig. 2.

The sample is homogeneous solid material and the gas is also homogeneous. The gas is fed in with a velocity V and temperature U .

At the right end point,

$$\frac{\partial u}{\partial x}(L, t) = 0.$$

At the left hand endpoint

$$Q = -k \frac{\partial u}{\partial x}(0, t)$$

which means that the heat source is held constant during the course of the experiment.

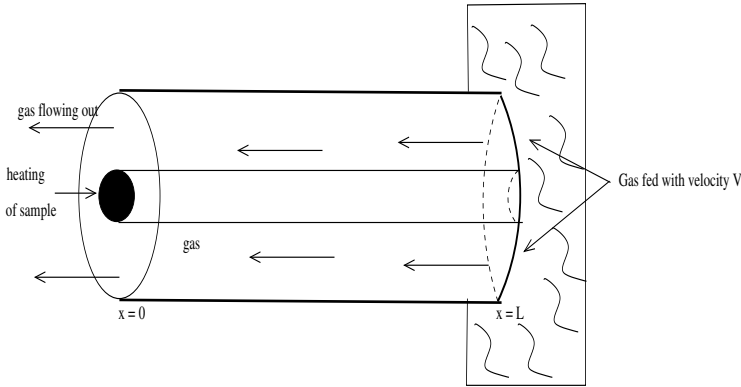


Figure 1: Geometry of the model problem.

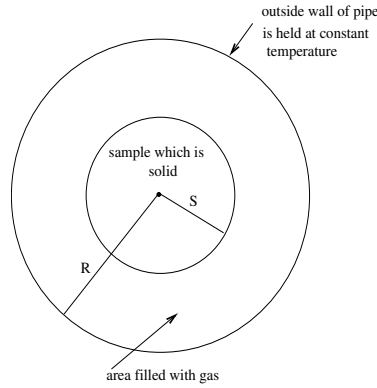


Figure 2: Cross section of the model.

Inside the sample, its temperature, $u(x, t)$ is governed by the heat equation. So, the problem of finding u involves solving

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t) &= a \frac{\partial^2 u}{\partial x^2}(x, t), & 0 < x < L, & \quad t > 0, \quad a = k/\rho c \\ u(x, 0) &= U, & 0 \leq x \leq L \\ \frac{\partial u}{\partial x}(L, t) &= 0, & t > 0 \\ \frac{\partial u}{\partial x}(0, t) &= -Q/k, & t > 0 \end{aligned}$$

where ρ , c , k , Q are known constants. ρ is the density, c is the specific heat of the substance, Q is the power of the source, k is the thermal conductivity and

a is the diffusivity. Note that we are making the assumption here about the gas, namely, the process is taking place sufficiently rapidly so that we can consider the density to be constant. This means that the temperature variations cannot be too large.

To change the boundary conditions to zero, we make the change of variables

$$\begin{aligned} f(x) &= \frac{(x-L)^2}{2L} \cdot \frac{Q}{k} \\ v(x, t) &= u(x, t) - f(x) \end{aligned} \quad (1)$$

to get the problem for $v(x, t)$

$$v_t(x, t) - av_{xx}(x, t) = \frac{Q}{c\rho L}, \quad 0 < x < L, \quad t > 0 \quad (2)$$

$$v_x(0, t) = 0, \quad v_x(L, t) = 0 \quad (3)$$

$$v(x, 0) = U - f(x) \quad (4)$$

Once $v(x, t)$ is known, the $u(x, t)$ is obtained from (1).

The associated eigenvalues and eigenfunctions for are

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{L} \right), \quad n = 0, 1, 2, \dots \\ e_n(x) &= \begin{cases} 1/2 & \text{if } n = 0 \\ \cos(\lambda_n x) & \text{if } n = 1, 2, \dots \end{cases} \end{aligned} \quad (5)$$

The eigenfunctions satisfy the orthogonality relations

$$\int_0^L e_n(x) e_p(x) dx = \begin{cases} 0, & n \neq p, \quad n, p = 0, 1, 2, \dots \\ L/4, & n = p = 0 \\ L/2, & n = p \geq 1 \end{cases} \quad (6)$$

Now we assume a solution for $v(x, t)$ in the form

$$v(x, t) = \sum_{n=0}^{\infty} \varphi_n(t) e_n(x) = \varphi_0(t) e_0(x) + \sum_{n=1}^{\infty} \varphi_n(t) e_n(x) \quad (7)$$

where the function $\varphi_n(t)$ are to be determined.

Recall that $e_0(x) = 1/2$ so

$$v_t - av_{xx} = \varphi'_0(t) \cdot 1/2 + \sum_{n=1}^{\infty} [\varphi'_n(t) + a\lambda_n^2 \varphi_n(t)] e_n(x) = \frac{Q}{c\rho L} \quad (8)$$

$$\begin{aligned}
 e'_n(0) &= e'_n(L) = 0 \quad \text{for } n = 0, 1, 2, \dots \\
 v(x, 0) &= \sum_{n=0}^{\infty} \varphi_n(0) e_n(x) = U - f(x)
 \end{aligned}$$

The problem now comes down to determining the $\varphi_n(t)$.

To find the $\varphi_n(0)$ multiply the equation by $e_p(x)$, integrate from 0 to L and apply the orthogonality relations to get

$$\sum_{n=0}^{\infty} \varphi_n(0) \int_0^L e_n(x) e_p(x) dx = \int_0^L [U - f(x)] e_p(x) dx \quad (9)$$

The left side is zero except when $p = n$. For

$p = 0$:

$$\varphi(0) = 2U - \frac{QL}{3k}$$

$p \geq 1$:

$$\frac{L}{2} [\varphi'_p(t) + a\lambda_p^2 \varphi_p(t)] = \int_0^L \frac{Q}{c\rho L} e_p(x) dx = 0$$

For $p = 0$:

$$\begin{aligned}
 \varphi'_0(t) &= \frac{4Q}{2c\rho L} \\
 \varphi_0(t) &= \frac{4Q}{2c\rho L} t + 2U - \frac{QL}{3k}
 \end{aligned}$$

$p \geq 1$

$$\varphi_p(0) = -\frac{2Q}{kL\lambda_p^2}$$

We summarize

$$\begin{aligned}
 \varphi_0(t) &= \frac{4Q}{2c\rho L} t + 2U - \frac{QL}{3k} \\
 \varphi_0(t) &= -\frac{2Q}{kL\lambda_p^2} e^{-a\lambda_n^2 t}
 \end{aligned}$$

Therefore the solution for $u(x, t)$ is:

$$u(x, t) = \frac{(x-L)^2 Q}{2Lk} + \frac{1}{2} \left(2U - \frac{QL}{3k} \right) + \frac{Q}{c\rho L} t + \sum_{n=1}^{\infty} \left(-\frac{2Q}{kL\lambda_p^2} \right) e^{-a\lambda_n^2 t} \cos(\lambda_n x) \quad (10)$$

3. The inverse problem

Now consider the same experiment. Suppose that k and c are unknown and $u(0, t)$ and $u(L, t)$ have been measured and so are known. Let $\varphi(t)$ denote the measured temperature value of the sample at $x = 0$ and $\psi(t)$ the measured temperature value of the sample at $x = L$.

Let us now suppose that we have values for k and c . u depends upon k and c and we write $u(x, t; k, c)$. If k and c are known, then the temperature values are given by Eqn. (10). We want to choose k and c so that the differences

$$\Delta(t, k, c) = \frac{LQ}{2k} + \frac{1}{2}\left(2U - \frac{QL}{3k}\right) + \frac{Q}{c\rho L}t + \sum_{n=1}^{\infty} \left(-\frac{2Q}{kL\lambda_n^2}\right) e^{-a\lambda_n^2 t} - \varphi(t) \quad (11)$$

$$\delta(t, k, c) = \frac{1}{2}\left(2U - \frac{QL}{3k}\right) + \frac{Q}{c\rho L}t + \sum_{n=1}^{\infty} \left(-\frac{2Q}{kL\lambda_n^2}\right) (-1)^n e^{-a\lambda_n^2 t} - \psi(t) \quad (12)$$

are close to zero. That we need to find k and c so that

$$h(k, c) = \left\{ \int_U^{T_{max}} [\Delta(t_i, k, c)^2 + \delta(t_i, k, c)^2] dt \right\}^{1/2} \quad (13)$$

is minimized. This determines k, c and consequently a .

4. Cooling of the Sample

Let us now take the same setup as before, and assume also that we do not know k and a . The sample will be warmed to a uniform temperature, U , and let us assume that we start heating and denote the power by Q . Both U and Q are known.

for $t > T$ the problem to be solved is

$$\begin{aligned} u_t(x, t) &= au_{xx}(x, t), & 0 < x < L, & \quad t > T \\ u_x(0, t) &= u_x(L, t) = 0, & t > T \\ u(x, T) &\text{ given} \end{aligned}$$

The solution is

$$u(x, t) = \frac{QT}{Lk} + U + \sum_{n=1}^{\infty} \frac{2Q}{kL\lambda_n^2} e^{-a\lambda_n^2(t-T)} \cos(\lambda_n x) \quad (14)$$

We now measure the temperature during the time of the experiment at some point. For definiteness, suppose we know the temperature at $x = L/2$ over the time interval $[0, T]$. Call these measured values $\theta(t)$.

The value, T , is also known. But by the time $t = T$, the part of the solution containing a will have, to all intents and purposes, disappeared. Thus, for $t > T$, that part of the solution will have no real effect on the solution beyond that.

The solution, therefore, shows there are two places where we can expect to get a value of k . First, the measured value will eventually become linear in t and we can get the slope and from slope, Q/kL , we can find k . But also we can find k by finding the limiting value of the temperature which is $QT/(kL) + U$. That value will probably be more accurate. To find a , we minimize the difference

$$\int_U^{T_{max}} [u(\frac{L}{2}, t) - \theta(t)]^2 dt \quad (15)$$

over a . This will give values for both k and a . $u(\frac{L}{2}, t)$ is given by

$$u(\frac{L}{2}, t) = \frac{QT}{Lk} + U + \sum_{n=1}^{\infty} \frac{2Q}{kL\lambda_n^2} e^{-a\lambda_n^2(t-T)} \cos(\frac{L\lambda_n}{2}) \quad (16)$$

5. Numerical results

In this section we will show some numerical results that determine the values of k and c and therefore $u(x, t)$ in the inverse problem. For the infinite series in equation (12) we took 100 terms to guarantee the convergence of the series. Since the values of $\varphi(t)$ and $\psi(t)$ are obtained from experiment, we used the trapezoidal rule to approximate the integral in Eqn. (13). The least squares method along with Newton's method for nonlinear systems are used for the minimization of Eqs. (13) and (15).

Example 1. In this example we consider an unknown sample given that the curve of the temperature at $x = 0$ and $x = L$, that is $u(0, t)$ and $u(L, t)$ are known from an experiment. The goal is to get the conductivity k and the specific heat c of the sample. Fig. 3 show the curve of $u(0, t)$ and $u(L, t)$ obtained from the experiment conducted in the time interval $[52, 155]$ seconds. Once k , c are found, the diffusivity a can be obtained from the relation $a = k/(\rho c)$.

By minimizing Eqn. (13), we found that $k = 33.47$ W/m-K and $c = 862.8$ W-s/Kg-K. The length of the sample is taken to be $L = 5$ cm., $Q = 1.4$ W/m², and $\rho = 3.98$ Kg/m³. The numerical results of the temperature distribution at the final time $T = 155$ sec. obtained from Eqn. (10) are shown in Fig. 4.

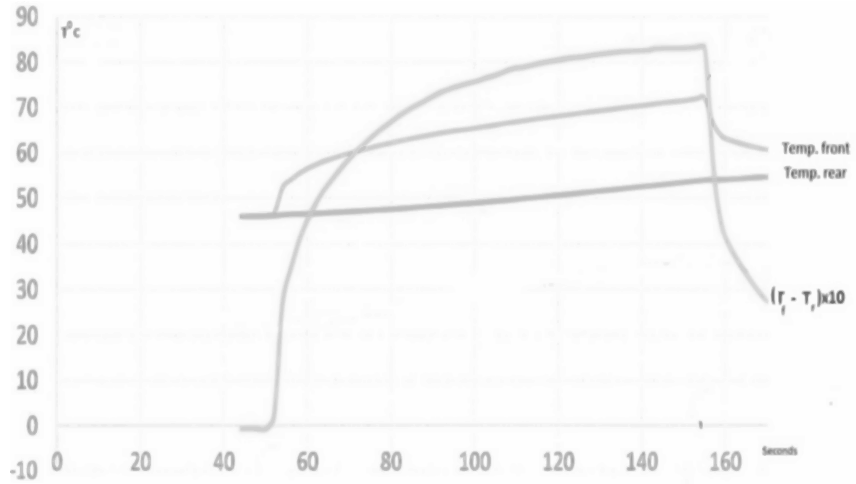


Figure 3: Experimental data: Example 1.

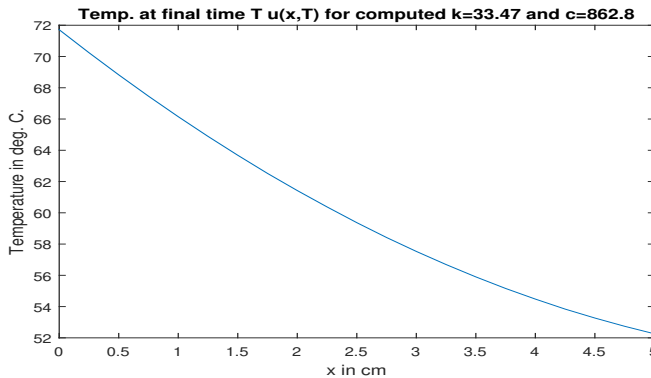


Figure 4: Temperature distribution at $t = 155$ sec..

Example 2. In this example we want to monitor the cooling of a known sample. We use as a sample the aluminum oxide whose properties are known. The experiment is conducted to get the values of the temperature at the middle of the sample, that is at $L/2$ during a period of time to be 120 seconds starting from an initial temperature of 20°C . The results of the experiment are shown

in Fig. 5. From the experiment one can observe that the limiting value l of the temperature is approximately $l = 132.5^{\circ}\text{C}$., this gives the approximate value of $k = QT/((l - U)L) = 30.135$ with $Q=1.4 \text{ W/m}^2$, $L=5 \text{ cm}$., and $U= 20^{\circ} \text{ C}$.. Given k , we minimize Eqn. 15 using again the least squares method along with Newton's method to get the approximate value of the diffusivity $a = 0.009321 \text{ m}^2/\text{s}$. The cooling of the sample 30 seconds after the experiment is shown in Fig. 6 obtained from eqn. 14.

From the example presented it can be seen that the numerical results are in full agreement with the properties of aluminum oxide sample.

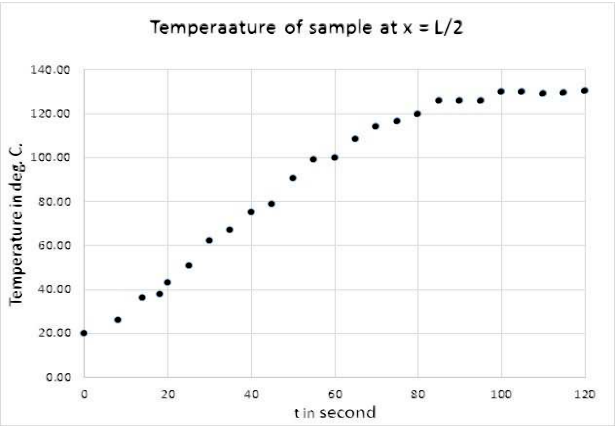


Figure 5: Experimental data curve for aluminum oxide at $x = L/2$.

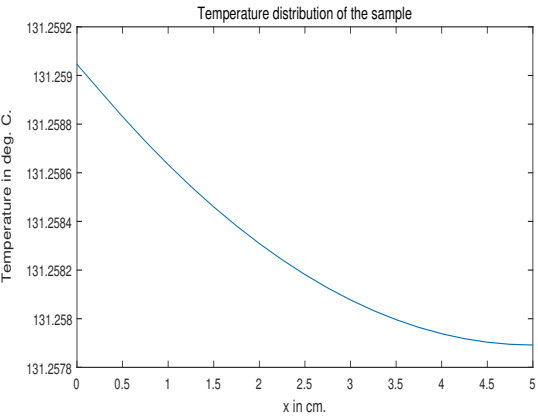


Figure 6: Temperature distribution of the sample $t = 30 \text{ sec.}$ after the experiment.

6. Conclusion

This paper deals with an inverse problem in thermodynamics. The present study shows that we can easily get the parameters of a sample that is completely insulated except at one boundary, from given experimental data of both boundaries of the problem. The model problem was presented and numerical results were given.

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7. Nomenclature

k = conductivity (W/m-K)
 c = specific heat (W-s/Kg-K)
 ρ = density (Kg/m³)
 Q = power (W/m²)
 $a = k/\rho c$ diffusivity (m²/s)
 U = initial temperature of sample (⁰ C.)

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