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FIXED-POINT AND BEST APPROXIMATION THEOREMS IN MODULAR SPACES

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Abstract: In this paper, we prove that two fixed point theorems for compact set-valued mappings in modular spaces. As application, a result on invariant best approximation is proved.

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1. Introduction and Preliminaries

The notion of modular spaces was introduced by Nakano [1] in 1950 as a generalization of metric spaces and then redefined and modified by Musielak and Ortiz [2] in 1959. Many results about fixed points in these spaces are considered such as, [3], [4], [5], and [6]. Recently Abed [7] defined the best approximation and proved results about proximinal set, Chebysev set and existence invariant best approximation. In the object of this note is to get Ky Fan type theorems on fixed point in complete modular spaces and used the second one to obtain invariant best approximation theorem. Recently, Abed and Abdul Sada [8] give a generalization of Dotson's fixed point theorem for non-expansive mappings

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on star-shaped subsets and then, in the setting of modular spaces and use it to prove a theorem of Brosowski-Meinaraus type on invariant approximation.

We denote 2^A, $CB(A)$ and $co(A)$ by the class of all nonempty subsets of A, the class of all closed bounded nonempty subsets of A and the convex set generated by A respectively.

Now, recall the following

Definition 1. *[6].*Let M be a linear space over $F(= RorC)$.

(i) $\gamma(v) = 0 \quad$ if and only if $v = 0$;

(ii) $\gamma(\alpha v) = \alpha(v) \quad$ for $\alpha \in F \quad with \quad |\alpha| = 1$, for all $v \in M$;

(iii) $\gamma(\alpha v + \beta u) \leq \gamma(v) + \gamma(u)$ if $\alpha, \beta \geq 0$, for all $,u, v \in M$.
if (iii) replaced by

(iii) $\gamma(\alpha v + \beta u) \leq \alpha\gamma(v) + \beta\gamma(u)$, for $\alpha, \beta \geq 0, \alpha + \beta = 1$ for all , $v, u \in M$, then M modular γ is called convex modular

Definition 2. *[6].* A modular γ defines a corresponding modular space, i.e., the space M_γ given by

$$M_\gamma = \{v \in M : \gamma(\alpha v) \to 0 \quad whenever \quad \alpha \to 0\}.$$

Remark 1.1. *[6] by condition (iii) above, if $u = 0$, then $\gamma(\alpha v) = \left(\gamma \frac{\alpha}{\beta} \beta v\right) \leq \gamma(\beta v)$, for all α, β in F, $0 \leq \alpha \leq \beta$. this shows that γ is increasing function.*

Definition 3. *[6].* The γ-ball, $B_r(u)$ centered at u $\in M_\gamma$ with radius $r > 0$ as
$B_r(u) = \{v \in M_\gamma; \gamma(u - v) < r\}$.
The class of all $\gamma-balls$, in modular space M_γ generates a topology which makes M_γ Hausdorff topological linear space. Every $\gamma - ball$ is convex set, therefore every modular space locally convex Hausdorff topological vector space *[7].*

Definition 4. *[6].* the $\gamma - distance$ between $v \in M_\gamma$ and $B \subset M_\gamma$ is $D_\gamma(v, B) =$inf$\{\gamma(v - u); u \in B\}$.

Definition 5. *[6].* Let M_γ be a modular space.

(a) A sequence $\{v_n\} \subset M_\gamma$ is said to be $\gamma-$convergent to $v \in M_\gamma$ and write $v_n \xrightarrow{\gamma}$ if $\gamma(v_n - v) \to 0$ as $n \to \infty$

(b) A sequence $\{v_n\}$ is called $\gamma-$ Cauchy whenever $\gamma(v_n) \to 0$ as, $\to \infty$.

(c) M_γ is called $\gamma-$ complete if any $\gamma-$ Cauchy sequence in M_γ is $\gamma-$ convergent.

(d) A subset $B \subset M_\gamma$ is called $\gamma-$ closed if for any sequence $\{v_n\} \subset \gamma-$ convergent to $v \in M_\gamma$, we have $v \in B$

(e) A $\gamma-$ closed subset $B \subset M_\gamma$ is called $\gamma-$ compact if any sequence $\{v_n\} \subset B$ has a $\gamma-$ convergent subsequence.

(f) A subset $B \subset M_\gamma$ is said to be $\gamma-$ bounded if $diam_\gamma(B) < \infty$, where $diam_\gamma(B) = sup\{\gamma(v-u); v, u \in B\}$ is called the $\gamma-$ diameter of B.

Let M_γ and $N_\rho$be two modular spaces, recall that a set -valued mapping $S : M_\gamma \to N_\rho$ is a subset of $M_\gamma \times N_\rho$; equivalently, S is a point to set mapping assigning to each $u \in M_\gamma$ a nonempty subset $S(u)$ of N_ρ. let $v \in M, v$ is called a fixed point of S if $v \in Sv$ (when S is a signal valued), v is fixed point of S if $v = Sv$.

Definition 6. A set-valued mapping S is upper semi continuous (shortly, u.s.c.) if and only if the set $\{u \in M_\gamma : S(x) \cap B \neq \phi\}$ is closed for each closed subset B of N_ρ.
Moreover, if N_ρ is a compact Hausdorff space, and if each value of S is closed, then S is $u.s.c.$ if and only if S has closed graph, i.e. , S is a closed subset of $M_\gamma \times N_\rho$.

Definition 7. Let M_γ be a modular space with modular function γ and $\phi \neq A \subset M_\gamma$ for $v \in M_\gamma$, $P_A(v)) = \{u \in A : \gamma(v-u) = D_\gamma(v, A)\}$ is the set of all best approximation of v by A and the multivalve mapping $P_A : M_\gamma \to 2^A$ is said to the metric projection on M_γ.

Definition 8. [7] Let M_γ be a modular space with modular function γ and $\phi \neq A \subset M_\gamma$ for $v \in M_\gamma, P_A(v) = \{u \in A : \gamma(v-u) = D_\gamma(v, A)\}$ is the set of all best approximation of v of A and the multivalve mapping $P_A : M_\gamma \to 2^A$ is said to the metric projection on M_γ.

2. Main Results

We begin with following fixed point theorems.

Theorem 9. *Let $A \neq \phi$ be a compact subset of modular space M_γ with modular function γ and $S : A \to CB(A)$ be an (u.s.c.) mapping with S(v)is convex for all v in some dense almost convex K of A. then S has a fixed point.*

Proof. for each $\epsilon > 0$, let

$$F_\epsilon = \{v \in A :\in S(v) + \overline{B}_\epsilon(0)\}$$

to prove the existence of fixed point of S it is sufficient (and necessary) to show $\cap F_\epsilon = \phi$. since for any $\epsilon > \delta, F_\epsilon \supset F_\delta$, it is sufficient, by the compactness of A, to show that each F_ϵ, is closed and nonempty.So let $\epsilon > 0$. Define the set-valued mappings

$$S_\epsilon : A \to 2^A, S_\epsilon(v) = (S + \overline{B}_\epsilon(0)) \cap A$$

and

$$R_\epsilon : A \to 2^A, R_\epsilon(v) = (v + \overline{B}_\epsilon(0)) \cap A, \quad \text{for} \quad v \in A$$

then $S_\epsilon = R_\epsilon \circ S, R\epsilon$ is a closed supset of $A \times A$ since
$R_\epsilon = \{(v,u) \in A \times A | u - v \in \overline{B}_\epsilon(0)\}$ and since $\overline{B}_\epsilon(0)$ is a closed subset of $A \times A$ and A is compact is follows that both $R\epsilon$ and S are (u.s.c.). Hence $S\epsilon$ is (u.s.c.). and S_ϵ is closed subset of $A \times A$.
let $\triangle$ be the diagonal in $A \times A$. Then
F_ϵ is the projection of compact set $\triangle \cap S\epsilon$ onto the domain of S_ϵ. it follows that F_ϵ is closed .
Now choose $z_1, ..., z_m \in K$ such that $K \subset \{z_i + \overline{B}_\epsilon(0) : 1 \leq i \leq m\}$ and $C = co\{z_1, ..., z_m\} \subset K$. Define $H_\epsilon \subset C \times C$ by $H_\epsilon = S_\epsilon \cap (C \times C)$. for each $v \in C, H_\epsilon(v)$ is closed, convex (since $C \subset A$) and nonempty (since $S_\epsilon + \overline{B}\epsilon$ contain some z_i).Moreover,H_ϵ is a closed subset of $C \times C$(since S_ϵ is closed). Thus H_ϵ has a fixed point by Kakutani's fixed point theorem ***[9]*** , say,u And u belong to F_ϵ, which is not empty.

□

Theorem 10. *: Let $A \neq \phi$ convex subset of complete modular space M_γ with modular function .Let $S : A \to CB(A)$ an (u.s.c.)such that S(v) is convex for all $v \in A$ and S(A) is contained in some compact subset C of A. Then S has fixed point.*

Proof. Let $B = coC$ and $K = \overline{B}$. Then K is compact,$B \subset A$ and $S(B) \subset C \subset B$.
Let $H = S \cap B \times B$. Then H is relatively closed subset of $B \times B$. Consider $\overline{H} \subset K \times K$ with closure relative to $K \times K$. Then H is a set-valued mapping from K to K, i.e., $\overline{H}^{-1}(K) = K$ since $\overline{H}^{-1}(K)$ is closure and contains B. Moreover $\overline{H}(K) \subset C \subset B$ and $= \overline{H} \cap (B \times B)$; so $\overline{H}(v) = H(v) = S(v)$ for all $v \in B$. Thus by Theorem 9 $\overline{H}$ has fixed point say v in K. But $v \in \overline{H}(v) \subset C \subset B$. so $v \in S(v)$. Hence S has fixed point. □

Theorem 11. *Let A be a compact convex subset of a convex modular M_γ and $f : A \to M_\gamma$ be a continuous function, then there exist a $u \in A$ such that*

$$\gamma(u - f(u)) = d_\gamma(f(u), A) \tag{1}$$

Proof. Let $i : A \to R^+$ be defined $i(v) = inf\{\gamma(u - v), u \in A\}$. Since f is continuous on A for each $v \in A$, then there exist a $u \in A$ such that $i(v) = \gamma(u - f(v))$ (because A is compact). Define a multivalve mapping $S : A \to 2^A$ by
$S(v) = \{u \in A : i(v) = \gamma(u - f(v))\} \subseteq$ then $S \neq \phi$ (as above). We will prove that

i. $S(v)$ is closed set;

ii. $S(v)$ is convex set;

iii S is $(u.s.c.)$.

For (i) suppose that z is an accumulation point of, then there exists a sequence $\langle z_n \rangle \subseteq S(v)$ such that $z_n \to z$. And we have

$$\gamma(z - f(v)) = \gamma\left(\lim_{n\to\infty} z_n - f(v)\right) = \lim_{n\to\infty} \gamma\left(z_n - f(v)\right) = i(v).$$

Thus z belong to $S(v)$, and then $S(v)$ is closed set.
For (ii), suppose that $0 \leq \lambda \leq 1$ and $u1, u2 \in S(v) \subset A$. Since A is convex, then $\lambda u1 + (1 - \lambda)u2 \in A$ and $D_\gamma(v, A) \leq \gamma(1 + (1 - \lambda)u2 - v)$
Now,

$$\begin{aligned} \gamma(\lambda u_1 + (1 - \lambda)u_2 - v) &\leq \lambda\gamma(u_1 - v) + (1 - \lambda)\gamma(u_2 - v) \\ &= i(v) \\ &= \gamma(\lambda u_1 + (1 - \lambda)u_2 - v) \end{aligned}$$

And this prove that S(v) is convex set.
For (iii),let C be a closed subset of A, we will prove that $S^{-1}(C) = \{u \in A : S(u) \cap C \neq \phi\}$ is closed subset of . Suppose that $v_0 \in A$ be an accumulation point $S^{-1}(C)$, then there exists a net $\langle v_a \rangle \subseteq S^{-1}(C)$ converg to v_0 . This implies that there is a net $u_a \in S(v_a) \cap C$. That is,$u_a \in C$ and $u_a\ inS(v_a)$ so, $\gamma\left(u_a - f(v_a)\right) = i(v_a)$ for each a. Since A is compact and C is closed subset of, then C is compact, so there is a $u_(0) \in C$ and a subnet $\langle u \rangle$ of $\langle u_a \rangle$ such that $u_\beta \to u_0$. Hence,$u_\beta \in S(v_a)$
$\Rightarrow \gamma\left(u_\beta - f(v_a)\right) = i(v)$, for each β.
$\Rightarrow \gamma\left(u_0 - f(v_0)\right) = i(v_0)$, which mean that $u_0 \in Sv_0 \cap C$. This implies that

$v_0 \in S^{-1}(C)$. Thus S is $(u.s.c.)$ set-valued mapping. Since A is compact and $S(A) \subset A$, then $S(A)$ is contained in compact set. Therefor by 10 there is a $u_0 \in A$ such that $u_0 \in Su_0$ that is

$$\gamma\left(u_0 - f(u_0)\right) = d\left(f(u_0), A\right).$$

To illustrate the utility of compactness condition in 11, we have the following

□

Example 2.1. Consider the unit ball $B_r(0)$ in modular space l^2 with convex modular function $\gamma(x) = \sqrt{(\sum_1^\infty |x_i|^2}$ where $x = (x_1, x_2, \cdots)$ and $|\Box|$ is absolute valued. $B_r(0)$ is closed and bounded but non-compact with topology induced by γ. For each x in $B_r(0)$,define the continuous function γ by

$$f(x) = \left(\sqrt{1 - ((x))^2}, x_1, x_2, .., x_n, ..\right)$$

Clearly, $\gamma(f(x)) = 1$ Suppose that f has a fixed point z, so, $\gamma(f(z)) = \gamma(z) = 1$ this implies that $z = 0$,i.e.,$\gamma(z) = 0$. which a contradiction,so (1) fails.

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