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ON THE EXPONENTIAL–GENERALIZED EXTENDED GOMPERTZ CUMULATIVE SIGMOID

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Abstract: In this paper we study the one–sided Hausdorff approximation of the shifted Heaviside step function by a family of Exponential–Generalized Extended Gompertz (EGEG) cumulative sigmoid. The model has a certain right of existence insofar as the theory of sigmoidal functions is well developed. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

Numerical examples are presented using *CAS MATHEMATICA*.

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Key Words: Exponential–Generalized–Extended Gompertz (EGEG) cumulative sigmoid, Heaviside step function, Hausdorff distance, upper and lower bounds

1. Introduction

The Gompertz model is well known and widely used in many aspects of biology. It has been frequently used to describe the growth of animals and plants, as well as the number or volume of bacteria and cancer cells. The model, referred to at the time as the Gompertz theoretical law of mortality, was first suggested and first applied Benjamin Gompertz in 1825 [1]. The Gompertz model is a special case of the four parameter Richards model and thus belongs to the Richards family of three–parameter sigmoidal growth models. The insurance industry quickly started to use his method of projecting death risk.

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The Gompertz and logistic curves are still used in industry, because these curves are well fitted to the cumulative number of faults observed in existing software development processes. Japanese software development companies prefer regression analysis based on deterministic functions such as Gompertz and Gompertz–Makeham–type curves to estimate the number of residual faults.

For some history, and new Gompertz model approach, see [2]–[11].

Some modifications, properties and applications of extended Gompertz and exponentiated extended Gompertz families of distributions can be found in [12]–[13].

In [14], the authors proposed a five–parameters exponentiated generalized extended Gompertz cumulative sigmoid:

$$M^*(t) = \left(1 - \left(1 - \left(1 - e^{-\frac{\beta}{\gamma}\left(e^{\gamma t}-1\right)}\right)^{\theta}\right)^{a}\right)^{b}, \quad (1)$$

where $t > 0$, $a > 0$, $b > 0$, $\theta > 0$, $\beta > 0$, $\gamma \geq 0$.

Definition 1. The *shifted Heaviside step function* is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

Definition 2. The Hausdorff distance [15] (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In this note we study the Hausdorff approximation of the *shifted Heaviside step function* by the family of type (1).

2. Main Results

We consider the following class of this family:

$$M(t) = \left(1 - \left(1 - \left(1 - e^{-\frac{\beta}{\gamma}\left(e^{\gamma t}-1\right)}\right)^{\theta}\right)^{a}\right)^{b} \tag{2}$$

with

$$t_0 = \frac{1}{\gamma}\ln\left(1 - \frac{\gamma}{\beta}\ln\left(1 - \left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)^{\frac{1}{\theta}}\right)\right); \quad M(t_0) = \frac{1}{2}. \tag{3}$$

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - ((2)–(3)) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{4}$$

The following theorem gives upper and lower bounds for d

Theorem. Let

$$\begin{aligned}
&p = -\tfrac{1}{2},\\
&q = 1 + ab\beta\theta\left(1 - \left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)^{\frac{1}{\theta}}\right)\\
&\times\left(1 - \tfrac{\gamma}{\beta}\ln\left(1 - \left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)^{\frac{1}{\theta}}\right)\right)\\
&\times\left(1 - \left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)^{\frac{\theta-1}{\theta}}\left(1 - 0.5^{\frac{1}{b}}\right)^{\frac{a-1}{a}} 0.5^{\frac{b-1}{b}}\\
&r = 2.1q.
\end{aligned} \tag{5}$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid ((2)–(3)) the following inequalities hold for $q > \dfrac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{6}$$

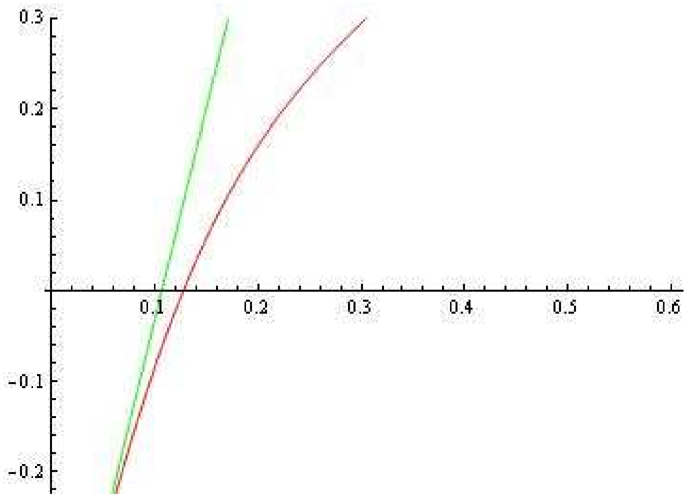

Figure 1: The functions $F(d)$ and $G(d)$ for $\beta = 2$; $\gamma = 3$; $a = 1.1$; $b = 8$; $\theta = 1.2$.

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{7}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{8}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).
In addition $G'(d) > 0$.

Further, for $q > \dfrac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

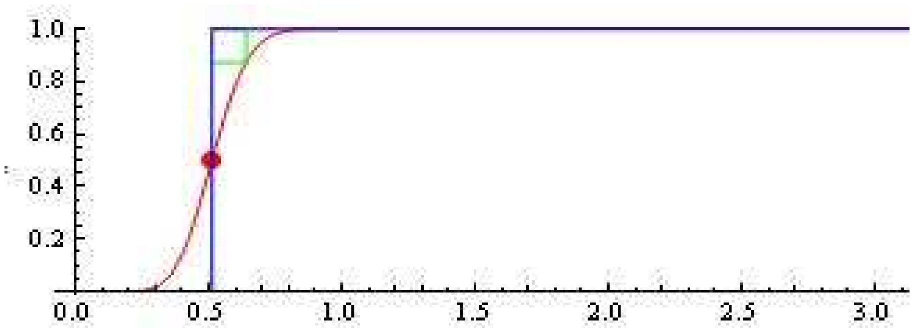

Figure 2: The model ((2)–(3)) for $\beta = 2$; $\gamma = 3$; $a = 1.1$; $b = 8$; $\theta = 1.2$, $t_0 = 0.512577$; H–distance $d = 0.127868$, $d_l = 0.101915$, $d_r = 0.232735$.

3. Numerical examples

The model ((2)–(3)) for $\beta = 2$; $\gamma = 3$; $a = 1.1$; $b = 8$; $\theta = 1.2$, $t_0 = 0.512577$ is visualized on Fig. 2.

From the nonlinear equation (4) and inequalities (6) we have: $d = 0.127868$, $d_l = 0.101915$, $d_r = 0.232735$.

The model ((2)–(3)) for $\beta = 3$; $\gamma = 5$; $a = 1.9$; $b = 10$; $\theta = 1.5$, $t_0 = 0.275937$ is visualized on Fig. 3.

From the nonlinear equation (4) and inequalities (6) we have: $d = 0.0782394$, $d_l = 0.0544249$, $d_r = 0.158427$.

4. Concluding Remarks

The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

The proposed new model can be successfully used to approximating data from Population Dynamic, Biostatistics and Debugging Theory.

For some approximation, computational and modelling aspects, see [16]–[31].

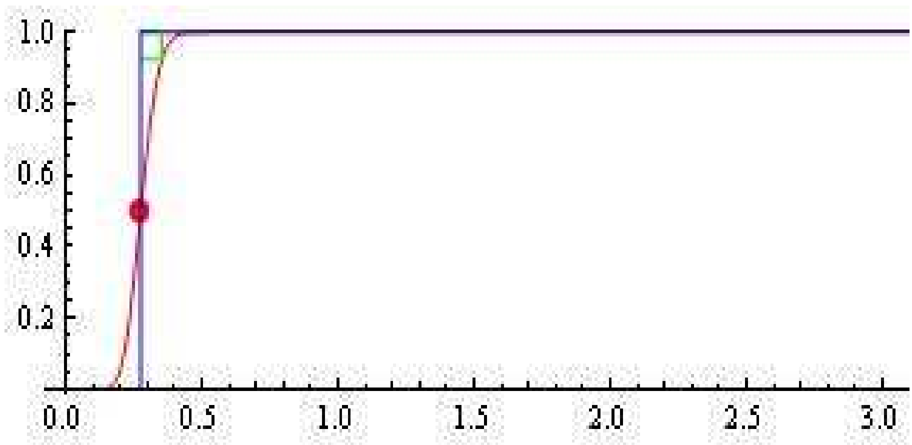

Figure 3: The model ((2)–(3)) for $\beta = 3$; $\gamma = 5$; $a = 1.9$; $b = 10$; $\theta = 1.5$, $t_0 = 0.275937$; H–distance $d = 0.0782394$, $d_l = 0.0544249$, $d_r = 0.158427$.

The results obtained in this paper can be used when controlling growth in Software Reliability Models, see [8], [9], [32].

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