COMPARISON AND CONVERGENCE THEOREMS FOR INITIAL-BOUNDARY VALUE PROBLEM TO ONE NONLINEAR PARABOLIC EQUATION

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Abstract: The difference scheme of the initial-boundary value problem for one nonlinear parabolic equation is considered. The convergence of the solution of the difference scheme to the solution of source problem is proved. For the same difference scheme the comparison theorem is proved and the uniqueness of the solution is obtained. The iteration process for finding the solution of difference scheme is constructed and its convergence is obtained.

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1. Introduction

In some mathematical models of diffusion process the system may be described as:

\[ U_t = a (x, t) U_{xx} + b (x, t) (U_x)^2 + f (x, t), \quad (x, t) \in \Omega \times (0, T], \]

(1)

\[ U (x, 0) = \varphi (x), \quad x \in \Omega, \]

(2)

\[ U (0, t) = \phi_0 (t), \quad U (1, t) = \phi_1 (t), \quad t \in (0, T], \]

(3)

where \( T \) is a positive constant, \( U = U (x, t) \) is solution, \( a, b, f, \varphi, \phi_0 \) and \( \phi_1 \) are given functions, \( \Omega = (0, 1) \).

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We assume that functions \( a = a(x, t), b = b(x, t), f = f(x, t) \) are continuous on \( \overline{\Omega} \times [0, T] \), \( \varphi \) is continuous on \( \overline{\Omega} \), \( \phi_0, \phi_1 \) are continuous on \([0, T]\) and

\[
a(x, t) > 0, \quad (x, t) \in \overline{\Omega} \times [0, T].
\]

The initial-boundary problem for the equation like (1) was considered in article [5] by means of finite-element approximation. The difference schemes to nonlinear parabolic equations were considered in a number of works (see, for example, [1], [2], [3], [4], [8]). This author received the same results as mentioned above for problem containing another kind of parabolic equation in articles [7], [9].

In the present work we investigate questions of approximate solution of the problem (1)-(3) using the difference scheme. In certain conditions the convergence of the solution of the difference scheme to the solution of source problem is proved. For the same difference scheme the comparison theorem is proved and the uniqueness of the solution is obtained. The iteration process for finding of the solution of difference scheme is constructed and in certain conditions its convergence is obtained.

2. Problem Formulation

Define a grid \( \omega_{h \tau} \) on the domain \( \overline{\Omega} \times [0, T] \) as follows:

\[
\omega_h = \{x_i = ih, \ h > 0, \ i = 0, 1, ..., M; \ hM = 1\},
\]
\[
\omega_\tau = \{t_j = j\tau, \ \tau > 0, \ j = 0, 1, ..., N; \ \tau N = T\},
\]

\[
\omega_{h \tau} = \omega_h \times \omega_\tau.
\]

Let \( y^j_i \) be a function defined on the grid \( \omega_{h \tau} \). Enter the following notations:

\[
\|y^j\|_{C(\omega_h)} = \max_{i=0,1,...,N} \|y^j_i\|, \quad \|y\|_{C(\omega_{h \tau})} = \max_{j=0,1,...,N} \|y^j\|_{C(\omega_h)}.
\]

Let

\[
g(u) = \frac{1}{2} \left(1 - \frac{1}{1 + u^2}\right). \quad (5)
\]

Note that

\[
0 \leq g(u) \leq \frac{1}{2}, \quad -1 < g'(u) < 1. \quad (6)
\]
Make the proper approximation of the term \((\partial U/\partial x)^2\):
\[
\left(\frac{\partial U}{\partial x}\right)^2 = \frac{2\lambda^2}{h^2} g \left(\frac{1}{2\lambda} \left[ U(x_{i+1}, t^j) - U(x_{i-1}, t^j) \right] \right) + O(h^2),
\]
where the value of \(\lambda\) can be chosen arbitrarily, therefore we will specify its value later.

Construct the difference scheme to the problem (1)-(3) as follows:
\[
u_{i}^{j+1} - u_i^j = a(x_i, t^{j+1}) \frac{1}{h^2} \left[ u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} \right] + \]
\[+ b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g \left( \frac{1}{2\lambda} \left[ u_{i+1}^{j+1} - u_i^{j+1} \right] \right) + f(x_i, t^{j+1}), \quad i = 1, 2, ..., M - 1, \quad j = 0, 1, ..., N - 1, \quad (7)
\]
\[u_0^j = \varphi_i, \quad i = 0, 1, ..., M, \quad (8)
\]
\[u_M^j = \phi_1^j, \quad j = 1, 2, ..., N, \quad (9)
\]
where \(\varphi_i = \varphi(x_i), i = 0, 1, ..., M, \phi_0^j = \phi_0(t^j), \phi_1^j = \phi_1(t^j), j = 1, 2, ..., N.

Further in this work we investigate the difference scheme (7)-(9). We prove the theorem of convergence of the solution of scheme (7)-(9) to the solution of the problem (1)-(3). We also prove the comparison theorem for the scheme (7)-(9), which implies the uniqueness of the solution of the scheme (7)-(9). Also, we construct the iteration process for finding of the solution of the scheme (7)-(9) and show its convergence to the solution of the scheme (7)-(9).

3. Problem Solution

In this section we will prove the theorems of comparison, uniqueness and convergence for the solution of the difference scheme (7)-(9). First we will refer to the two important theorems from book [6], pages 15-17.

Consider the grid function \(y_i\) defined on the grid \(\omega_h\). Define the maximum norm on the grid \(\omega_h\) as follows
\[\|y\|_{C(\omega_h)} = \max_{i=0,1,...,M} |y_i|.
\]
Let the grid function \(y_i\) satisfy the following conditions:
\[A[y_i] = A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, ..., N - 1, \quad (10)
\]
\[y_0 = \mu_1, \quad y_N = \mu_2, \quad (11)
\]
where \(A_i, B_i, C_i, F_i\) are constants.
Theorem 1. (see [6]) Let the conditions

\[ A_i > 0, \ B_i > 0, \ D_i = C_i - A_i - B_i \geq 0, \]

be fulfilled for all \( i = 1, 2, \ldots, M - 1 \) and the grid function \( y_i \), which is not constant identically, satisfy at all the inner nodes \( i = 1, 2, \ldots, M - 1 \) the condition \( \Lambda [y_i] \leq 0 \). Then \( y_i \) cannot take the minimal negative value at the inner points, that is, for \( i = 1, 2, \ldots, M - 1 \). In particular, if \( \mu_1 \geq 0 \) and \( \mu_2 \geq 0 \), then \( y_i \geq 0 \) for all \( i = 0, 1, \ldots, M \).

Theorem 2. (see [6]) Let the conditions

\[ |A_i| > 0, \ |B_i| > 0, \ D_i = |C_i| - |A_i| - |B_i| > 0 \]

hold for all \( i = 1, 2, \ldots, M - 1 \) and \( \mu_1 = \mu_2 = 0 \). A solution of the problem (10), (11) admits the estimate

\[ \|y\|_{C(\omega_h)} \leq \max_{i=1,2,\ldots,M-1} \left| \frac{F_i}{D_i} \right|. \]

With Theorem 1 and Theorem 2 we can then prove our result:

Theorem 3. (Comparison Theorem) Let \( \bar{u} \) and \( \bar{u} \) be functions defined on the grid \( \omega_{h_T} \), satisfying the following inequalities:

\[ \frac{\bar{u}_{i+1}^j - \bar{u}_i^j}{\tau} - a(x_i, t^{j+1}) \frac{1}{h^2} \left[ \bar{u}_{i+1}^{j+1} - 2\bar{u}_i^{j+1} + \bar{u}_{i-1}^{j+1} \right] - \]

\[ - \frac{b(x_i, t^{j+1}) 2\lambda^2}{h^2} g \left( \frac{1}{2\lambda} \left[ \bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1} \right] \right) \geq \]

\[ \geq \frac{\bar{u}_{i+1}^j - \bar{u}_i^j}{\tau} - a(x_i, t^{j+1}) \frac{1}{h^2} \left[ \bar{u}_{i+1}^{j+1} - 2\bar{u}_i^{j+1} + \bar{u}_{i-1}^{j+1} \right] - \]

\[ - \frac{b(x_i, t^{j+1}) 2\lambda^2}{h^2} g \left( \frac{1}{2\lambda} \left[ \bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1} \right] \right), \]

\[ i = 1, 2, \ldots, M - 1, \quad j = 0, 1, \ldots, N - 1, \quad (12) \]

\[ \bar{u}_i^0 \geq \bar{u}_i^0, \quad i = 0, 1, \ldots, M, \quad (13) \]

\[ \bar{u}_0^j \geq \bar{u}_0^j, \quad \bar{u}_M^j \geq \bar{u}_M^j, \quad j = 1, 2, \ldots, N. \quad (14) \]

If \( \lambda \) satisfies the following condition

\[ 0 < \lambda < \frac{\max_{(x, t) \in \Omega \times [0, T]} \{|b(x, t)|\}}{\min_{(x, t) \in \Omega \times [0, T]} \{a(x, t)\}}, \quad (15) \]

then \( \bar{u}_i^j \geq \bar{u}_i^j \) on the grid \( \omega_{h_T} \).
Proof. Enter the following notations

\[ y^j_i = \bar{u}^j_i - \bar{u}^j_i, \quad i = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, N. \]

Then the inequalities (12) can be written as follows:

\[
\frac{y^j_{i+1} - y^j_i}{\tau} - a(x_i, t^{j+1}) \frac{1}{\lambda h^2} \left[ y^j_{i+1} - 2y^j_i + y^j_{i-1} \right] \geq
\]

\[
\geq b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[ g \left( \frac{1}{2\lambda} \left[ \bar{u}^j_{i+1} - \bar{u}^j_{i-1} \right] \right) - g \left( \frac{1}{2\lambda} \left[ \bar{u}^j_{i+1} - \bar{u}^j_{i-1} \right] \right) \right],
\]

where \( \theta^j_i \) are numbers from the interval \((0, 1)\) according to the multidimensional scheme of the Lagrange Mean Value Theorem.

Denote by \( A^j_i \) and \( B^j_i \) the following quantities:

\[
A^j_i = \frac{1}{h^2} \left[ a(x_i, t^{j+1}) + \lambda b(x_i, t^{j+1}) \times
\right.
\]

\[
\times g' \left( \frac{1}{2\lambda} \left[ \theta^j_i \bar{u}^j_{i+1} + \left( 1 - \theta^j_i \right) \bar{u}^j_{i+1} - \theta^j_i \bar{u}^j_{i-1} - \left( 1 - \theta^j_i \right) \bar{u}^j_{i-1} \right] \right),
\]

\[
B^j_i = \frac{1}{h^2} \left[ a(x_i, t^{j+1}) - \lambda b(x_i, t^{j+1}) \times
\right.
\]

\[
\times g' \left( \frac{1}{2\lambda} \left[ \theta^j_i \bar{u}^j_{i+1} + \left( 1 - \theta^j_i \right) \bar{u}^j_{i+1} - \theta^j_i \bar{u}^j_{i-1} - \left( 1 - \theta^j_i \right) \bar{u}^j_{i-1} \right] \right),
\]

Then (16) can be written as follows:

\[
A^j_i y^j_{i+1} - \left( A^j_i + B^j_i + \frac{1}{\tau} \right) y^j_i + B^j_i y^j_{i-1} \leq -\frac{y^j_i}{\tau},
\]
Applying the method of mathematical induction on index \( j \) we prove, that
\[ y^i_j \geq 0, \quad i = 0, 1, ..., M, \quad j = 0, 1, ..., N. \tag{18} \]

From condition (13) when \( j = 0 \), we have
\[ y^0_0 \geq 0, \quad i = 0, 1, ..., M. \]

Suppose, that (18) is valid when \( j = l - 1 \) and then we will show its validity for \( j = l \). Write (17) as follows:
\[ A^l_i y^l_{i+1} - \left( A^l_i + B^l_i + \frac{1}{\tau} \right) y^l_i + B^l_i y^{l-1}_i \leq -\frac{y^{l-1}_i}{\tau}, \quad i = 0, 1, ..., M. \tag{19} \]

According to the conditions (4), (6), (15) \( A^l_i > 0, \quad B^l_i > 0, \quad A^l_i + B^l_i + \frac{1}{\tau} > 0 \). Taking into account inequalities (14), we have \( y^0_0 \geq 0, \quad y^M_M \geq 0 \). As for the right side of (19), it is nonpositive. Therefore we can apply the Theorem 1 to \( y^l_i \), and we obtain \( y^l_i \geq 0, \quad i = 0, 1, ..., M \). The Theorem 3 is proved. \( \square \)

**Theorem 4.** (Uniqueness) Let \( \lambda \) satisfy the following condition
\[
0 < \lambda < \min_{(x, t) \in \Omega \times [0, T]} \left\{ a(x, t) \right\} \quad \text{and} \quad \max_{(x, t) \in \Omega \times [0, T]} \left\{ |b(x, t)| \right\}.
\]

If the solution of the difference scheme (7)-(9) exists, it is unique.

The proof is simply obtained from the Theorem 3.

**Theorem 5.** (Convergence of Solution) Let there exist a solution \( U \) of the problem (1)-(3) such that \( \left| \frac{\partial^4 U}{\partial x^4} \right| \leq V, \quad \left| \frac{\partial^2 U}{\partial t^2} \right| \leq V \) for some positive constant \( V \).

If \( \lambda \) satisfies the following condition
\[
0 < \lambda < \min_{(x, t) \in \Omega \times [0, T]} \left\{ a(x, t) \right\} \quad \text{and} \quad \max_{(x, t) \in \Omega \times [0, T]} \left\{ |b(x, t)| \right\}
\]

and the solution \( u \) of the difference scheme (7)-(9) exists for small values of \( \tau \) and \( h \) then the following estimate takes place
\[
\| u - U \|_{C(\omega, \tau)} \leq TW \left( \tau + h^2 \right),
\]
where \( W \) is positive constant independent from \( \tau \) and \( h \).
Proof. For the difference equations (7) we have:

$$
\frac{U_i^{j+1} - U_i^j}{\tau} = a(x_i, t^{j+1}) \frac{1}{h^2} \left[ U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1} \right] +
$$

$$
+ b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g \left( \frac{1}{2\lambda} \left[ U_{i+1}^{j+1} - U_i^{j+1} \right] \right) + f(x_i, t^{j+1}) + O(\tau + h^2),
$$

where $Q(\tau + h^2)$ is a quantity, which can be expressed as follows

$$
Q(\tau + h^2) = (\tau + h^2) q(\tau, h)
$$

for some bounded function $q$.

The grid function $Y_i^j = u_i^j - U_i^j$, is the solution of the following equations:

$$
\frac{Y_i^{j+1} - Y_i^j}{\tau} = a(x_i, t^{j+1}) \frac{1}{h^2} \left[ Y_{i+1}^{j+1} - 2Y_i^{j+1} + Y_{i-1}^{j+1} \right] + O(\tau + h^2) =
$$

$$
= b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[ g \left( \frac{1}{2\lambda} \left[ u_{i+1}^{j+1} - u_i^{j+1} \right] \right) - g \left( \frac{1}{2\lambda} \left[ U_{i+1}^{j+1} - U_i^{j+1} \right] \right) \right]
$$

$$
i = 1, 2, ..., M-1, \quad j = 0, 1, ..., N-1.
$$

(21)

Transform the right side of the inequality (21) using Lagrange Mean Value Theorem:

$$
b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[ g \left( \frac{1}{2\lambda} \left[ u_{i+1}^{j+1} - u_i^{j+1} \right] \right) - g \left( \frac{1}{2\lambda} \left[ U_{i+1}^{j+1} - U_i^{j+1} \right] \right) \right] =
$$

$$
= \frac{\lambda}{h^2} b(x_i, t^{j+1}) \left( Y_{i+1}^{j+1} - Y_i^{j+1} \right) \times
g' \left( \frac{1}{2\lambda} \left[ \theta_i^{j+1} u_{i+1}^{j+1} + (1 - \theta_i^{j+1}) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - (1 - \theta_i^{j+1}) U_{i-1}^{j+1} \right] \right),
$$

where $\theta_i^{j+1}$ are numbers from the interval $(0, 1)$ according to the multidimensional scheme of Lagrange Mean Value Theorem.

Denote by $A_i^j$ and $B_i^j$ the following quantities:

$$
A_i^{j+1} = \frac{1}{h^2} \left[ a(x_i, t^{j+1}) + \lambda b(x_i, t^{j+1}) \times
g' \left( \frac{1}{2\lambda} \left[ \theta_i^{j+1} u_{i+1}^{j+1} + (1 - \theta_i^{j+1}) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - (1 - \theta_i^{j+1}) U_{i-1}^{j+1} \right] \right) \right],
$$
$$B_i^{j+1} = \frac{1}{h^2} \left[ a(x_i, t^{j+1}) - \lambda b(x_i, t^{j+1}) \times \right.$$\
$$\left. \times g'\left( \frac{1}{2\lambda} \left[ \theta_i^{j+1} u_{i+1}^{j+1} + (1 - \theta_i^{j+1}) u_{i+1}^{j+1} \right] - \theta_i^{j+1} u_i^{j+1} - (1 - \theta_i^{j+1}) U_i^{j+1} \right) \right] ,$$

Then equation (21) can be written as follows

$$A_i^{j+1} Y_{i+1}^{j+1} - \left( A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} \right) Y_i^{j+1} + B_i^{j+1} Y_{i-1}^{j+1} = -\frac{Y_i^j}{\tau} + O(\tau + h^2) ,$$

$$i = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, N. \quad (22)$$

According to the conditions (4),(6),(20) \( A_i^{j+1} \geq 0, B_i^{j+1} \geq 0, A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} > 0, \quad \left| A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} \right| - \left| A_i^{j+1} \right| - \left| B_i^{j+1} \right| > 0. \) Therefore we can apply the theorem 2 to (22). We obtain

$$\| Y_i^{j+1} \|_{C(\omega_h)} \leq \tau \left( \frac{1}{\tau} \| Y_i^j \|_{C(\omega_h)} + W(\tau + h^2) \right) , \quad j = 0, 1, \ldots, N-1 .$$

Taking into account that \( \| Y_0^0 \|_{C(\omega_h)} = 0, \) we have

$$\| Y_i^{j+1} \|_{C(\omega_h)} \leq \| Y_i^j \|_{C(\omega_h)} + \tau W(\tau + h^2) \leq \| Y_i^{j-1} \|_{C(\omega_h)} + 2\tau W(\tau + h^2) \leq$$

$$\leq \| Y_i^{j-2} \|_{C(\omega_h)} + 3\tau W(\tau + h^2) \leq \| Y_0 \|_{C(\omega_h)} + (j + 1) \tau W(\tau + h^2) \leq$$

$$\leq TW(\tau + h^2) , \quad j = 0, 1, \ldots, N-1 .$$

The theorem 5 is proved.

4. Convergence of Iteration Scheme

In this section we construct the iteration process for finding of the solution of the difference scheme (7)-(9).

$$u_i^{j+1,l+1} = (1 - \sigma) u_i^{j+1,l} + \sigma \left\{ \tau \left[ a(x_i, t^{j+1}) \frac{1}{h^2} \left[ u_i^{j+1,l} - 2u_i^{j+1,l} + u_i^{j+1,l} \right] + \right. \right.$$\
$$\left. + b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[ \frac{1}{2\lambda} \left[ u_i^{j+1,l} - u_i^{j+1,l} \right] \right] + f(x_i, t^{j+1}) \right] + u_i^{j+1,l} \right\} ,$$

$$l = 0, 1, \ldots; \quad i = 1, 2, \ldots, M - 1, \quad j = 0, 1, \ldots, N - 1 , \quad (23)$$

$$u_i^{j+1,0} = u_i^j , \quad i = 0, 1, \ldots, M , \quad (24)$$

$$u_1^{j+1,l} = \phi_1^{j+1} , \quad u_M^{j+1,l} = \phi_1^{j+1} , \quad l = 1, 2, \ldots; \quad j = 0, 1, \ldots, N - 1 . \quad (25)$$
**Theorem 6.** (Convergence of Iteration Scheme) Let $\lambda$ satisfies the following condition

$$0 < \lambda < \frac{\min_{(x, t) \in \Omega \times [0, T]} \{a(x, t)\}}{\max_{(x, t) \in \Omega \times [0, T]} \{|b(x, t)|\}},$$

and $\sigma$ satisfies the following condition

$$0 < \sigma \leq \frac{h^2}{h^2 + 2\tau \max_{(x, t) \in \Omega \times [0, T]} \{a(x, t)\}}$$

then iteration process (23)-(25) converges for each $j$ and the limit is the solution of the difference scheme (7)-(9).

**Proof.** It is clear, that if the $\lim_{l \to \infty} \left( u_0^{j+1,l}, u_1^{j+1,l}, \ldots, u_M^{j+1,l} \right)$ exists then it will be the solution of the difference scheme (7)-(9).

Consider $u_i^{j+1,l+1} - u_i^{j+1,l}$. Taking into account the method, used in the proof of theorem 3, we have:

$$u_i^{j+1,l+1} - u_i^{j+1,l} = (1 - \sigma) \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right)$$

$$+ \sigma \tau \left\{ a(x_i, t_j^{1}) \frac{1}{h^2} \left[ \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) - 2 \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) \right] + \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) \right\}$$

$$+ b(x_i, t_j^{1+1}) \frac{2\lambda^2}{h^2} \left\{ g \left( \frac{1}{2\lambda} \left[ u_i^{j+1,l} - u_i^{j+1,l-1} \right] \right) - g \left( \frac{1}{2\lambda} \left[ u_i^{j+1,l-1} - u_i^{j+1,l-1} \right] \right) \right\}$$

$$= (1 - \sigma) \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right)$$

$$+ \sigma \tau \left\{ a(x_i, t_j^{1}) \left[ \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) - 2 \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) \right] + \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) \right\}$$

$$+ \lambda b(x_i, t_j^{1+1}) \left\{ \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right) - \left( u_i^{j+1,l-1} - u_i^{j+1,l-1} \right) \right\}$$

$$\times \left( \frac{1}{2\lambda} \left[ \theta_i^{j+1,l} u_i^{j+1,l} + (1 - \theta_i^{j+1,l}) u_i^{j+1,l-1} - \theta_i^{j+1,l} u_i^{j+1,l-1} \right) \right)$$

$$\left( 1 + \theta_i^{j+1,l} u_i^{j+1,l-1} \right)$$

$$= \left\{ 1 - \sigma - \frac{\sigma \tau}{h^2} \left[ A_i^{j+1,l} + B_i^{j+1,l} \right] \right\} \left( u_i^{j+1,l} - u_i^{j+1,l-1} \right)$$
Apply the maximum norm to both sides of the last equation:

\[ \| u^{j+1,l+1} - u^{j+1,l} \|_{C(\omega_h)} = \max_{i=1,2,\ldots,M-1} \left| \frac{\sigma \tau A_i^{j+1,l}}{h^2} (u_{i+1}^{j+1,l} - u_{i+1}^{j+1,l-1}) + \frac{\sigma \tau B_i^{j+1,l}}{h^2} (u_i^{j+1,l} - u_i^{j+1,l-1}) \right| + \left| \frac{\sigma \tau A_i^{j+1,l}}{h^2} (u_i^{j+1,l} - u_i^{j+1,l-1}) \right| \leq \max_{i=1,2,\ldots,M-1} \left( 1 - \sigma - \frac{\sigma \tau}{h^2} \left[ A_i^{j+1,l} + B_i^{j+1,l} \right] \right) \left( \| u^{j+1,l} - u^{j+1,l-1} \|_{C(\omega_h)} + \left\| \frac{\sigma \tau A_i^{j+1,l}}{h^2} + \frac{\sigma \tau B_i^{j+1,l}}{h^2} \right\| \right) \right| \left( \| u^{j+1,l} - u^{j+1,l-1} \|_{C(\omega_h)} \right) \tag{28} \]

(4), (6), (26) imply that

\[ 0 < A_i^{j+1,l} \leq 2 \max_{(x, t) \in \Omega \times [0, T]} \{ a(x, t) \}, \]

\[ 0 < B_i^{j+1,l} \leq 2 \max_{(x, t) \in \Omega \times [0, T]} \{ a(x, t) \}. \]

According to (27) we have

\[ 1 - \sigma - \frac{\sigma \tau}{h^2} \left[ A_i^{j+1,l} + B_i^{j+1,l} \right] \geq 0, \]
therefore we can remove the module in the right part of (28) and we have:
\[ \left\| u_{j+1,l+1}^{j+1} - u_{j+1,l}^{j+1} \right\|_{C(\omega_h)} \leq (1 - \sigma) \left\| u_{j+1,l}^{j+1} - u_{j+1,l-1}^{j+1} \right\|_{C(\omega_h)}. \]

It is clear that \(0 < (1 - \sigma) < 1\), therefore the sequence \(\left( u_{j+1,l}^{j+1} \right)_{l=1}^{\infty}\) converges and its limit is the solution of the difference scheme (7)-(9).

\[ \square \]

5. Conclusion

The results obtained in this work allow to get numerical solution of the problem like (1)-(3). Numerical experiments were conducted which confirm the validity of the theorems proved above.

References


