ON THE MINIMALITY AND TOTAL DEVELOPABILITY OF THE TIME-LIKE RULED SURFACES WITH THE TIME-LIKE GENERATING SPACE IN THE MINKOWSKI SPACE $\mathbb{I}R^n_1$

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Abstract: The purpose of this paper is, first, to introduce a summary of known results and the definition of the time-like ruled surface with the time-like generating space in the Minkowski space $\mathbb{I}R^n_1$ (Section 1); second, to present some characteristic results related with minimality and total developability of the ruled surface in the Minkowski space $\mathbb{I}R^n_1$ (Section 2).

AMS Subject Classification: 53A10, 53A35

Key Words: Minkowski space, time-like ruled surface, minimality, total developability

1. Introduction

We will assume throughout that this paper that all manifolds, maps, vector fields, etc. are differentiable of class $C^\infty$.

First of all, we give some properties of a general submanifold $M$ of the Minkowski $n-$space $\mathbb{I}R^n_1$, [2]. Let $\overline{D}$ be a Levi-Civita connection of $\mathbb{I}R^n_1$ and $D$ be a Levi-Civita connection of $M$. If $X, Y \in \chi(M)$ and $V$ is the second fundamental tensor of $M$, we have by decomposing $\overline{D}_X Y$
in a tangential and a normal component
\[ D_X Y = D_X Y + V(X, Y) \]  \hspace{1cm} (1.1)

The equation (1.1) is called the **Gauss Equation**.

If \( \zeta \) is any normal vector field on \( M \), we find the **Weingarten Equation** by decomposing \( D_X \zeta \) in a tangential component and a normal component as
\[ D_X \zeta = -A_\zeta(X) + D_{\chi} \zeta. \]  \hspace{1cm} (1.2)

\( A_\zeta \) determines at each point a self-adjoint linear map and \( D_{\chi} \) is a metric connection in the normal bundle \( \chi(\chi(M)). \) We note that, in this paper, \( A_\zeta \) will be used for the linear map and the corresponding matrix of the linear map.

If the metric tensor of \( IR^n_1 \) is denoted by \(<,>\), from the equation (1.1) and (1.2), it follows
\[ < V(X, Y), \zeta > = < A_\zeta(X), Y >. \]  \hspace{1cm} (1.3)

If \( \zeta_1, \zeta_2, ..., \zeta_{n-m} \) constitute an orthonormal basis of \( \chi(\chi(M)) \), then we set
\[ V(X, Y) = \sum_{j=1}^{n-m} < A_\zeta(X), Y > \zeta_j. \]  \hspace{1cm} (1.4)

The **mean curvature** \( H \) of \( M \) at the point \( P \) is given by
\[ H = \sum_{j=1}^{n-m} \frac{\text{tr}A_\zeta j}{\dim\chi_j}. \]  \hspace{1cm} (1.5)

For every \( X_i \in \chi(M), \ 1 \leq i \leq 4 \) the 4th order covariant tensor field defined by \( R \) as
\[ R(X_1, X_2, X_3, X_4) = < X_1, R(X_3, X_4)X_2 > \]
is called the **Riemannian curvature tensor field** and its value at a point \( P \in M \) is called the **Riemannian curvature** of \( M \) at the point \( P \).
If $V$ is the second fundamental tensor, then we have
\[
< Y, R(X,Y)X > = < V(X,X), V(Y,Y) > - < V(X,Y), V(X,Y) > .
\]
(1.6)

Let $\Pi$ be a tangent plane of $M$ at $P$. For all $X_P, Y_P \in \Pi$, the real function $K$ defined by
\[
\]
(1.7)
is called the sectional curvature function. $K(X_P, Y_P)$ is called the sectional curvature of $M$ at $P$.

Let $R$ be the Riemann curvature tensor and $\{e_1, e_2, ..., e_m\}$ be a system of orthonormal basis of $T_M(P)$. The tensor field $S$, defined in the form
\[
S(X, Y) = \sum_{i=1}^{m} \epsilon_i < R(X, e_i)Y, e_i >,
\]
(1.8)
is called the Ricci curvature tensor field and the value of $S(X, Y)$ at $P \in M$ is called the Ricci curvature, where
\[
\epsilon_i = < e_i, e_i >, \epsilon_i = \begin{cases} 
-1 & \text{if } e_i \text{ time-like,} \\
1 & \text{if } e_i \text{ space-like.}
\end{cases}
\]

The real number $r_{sk}$, defined in the form
\[
r_{sk} = \sum_{i \neq j} K(e_i, e_j) = 2 \sum_{i<j} K(e_i, e_j),
\]
(1.9)
is called the scalar curvature tensor field of $M$.

Let $V$ be the second fundamental tensor of $M$. If
\[
V(X, X) = 0,
\]
(1.10)
for $X \in \chi(M)$, then $X$ called asymptotic vector field on $M$. If
\[
V(X, Y) = 0,
\]
(1.11)
for all \( X, Y \in \chi(M) \), then \( M \) is *totally geodesic*.

Let \( M \) be a \((k+1)\)-dimensional ruled surface in \( IR^m_1 \). Then \( M \) can be locally represented by

\[
\phi(s, u_1, u_2, ..., u_k) = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s), \quad u_i \in IR, \ 1 \leq i \leq k. \quad (1.12)
\]

If the generating space \( E_k(s) = sp\{e_1, e_2, ..., e_k\} \) of \( M \) will be assumed a *time-like subspace* and the base curve \( \alpha \) is *space-like", then this surface is called the \((k+1)\)-dimensional *time-like ruled surface* in \( IR^m_1 \), [1].

If

\[
\text{rank}[e_0, e_1, ..., e_k, \overline{D}_e_0 e_1, ..., \overline{D}_e_0 e_k] = 2k - m, \quad (1.13)
\]

at each point \( P \) of \( M \), then \( M \) will be called as *m-developable*. If \( m = -1 \), then the time-like generalized ruled surface \( M \) is called as *non-developable*. If \( m = k - 1 \), \( M \) is called as *total developable*, where \( e_0 \) is the tangent vector of the base curve.

Suppose that \( \{e_0, e_1, ..., e_k\} \) is an orthonormal base field of the tangential bundle \( \chi(M) \) and \( \{\zeta_1, \zeta_2, ..., \zeta_{n-k-1}\} \) an orthonormal base field of the normal bundle \( \chi^\perp(M) \). Then an orthonormal base field of \( \chi(IR^m_1) \) is \( \{e_0, e_1, ..., e_k, \zeta_1, ..., \zeta_{n-k-1}\} \).

If we write the Weingarten derivative equation for this base vectors we have,

\[
\overline{D}_{e_i} \zeta_j = -A_{ij} (e_i) + D^\perp_{e_i} \zeta_j, \quad (1.14)
\]

or

\[
\begin{align*}
\overline{D}_{e_0} \zeta_j &= a^{j0} e_0 + \sum_{r=1}^{k} a^{jr}_{0r} e_r + \sum_{s=1}^{n-k-1} b^j_{0s} \zeta_s, \\
&1 \leq j \leq n - k - 1, \\
\overline{D}_{e_i} \zeta_j &= a^{ji} e_0 + \sum_{r=1}^{k} a^{jr}_{ir} e_r + \sum_{s=1}^{n-k-1} b^j_{is} \zeta_s, \quad 1 \leq i \leq k. \quad (1.15)
\end{align*}
\]
From the above derivative equation we have

\[
A_{\zeta_j} = - \begin{bmatrix}
    a_{j0} & a_{j1} & \cdots & a_{jk} \\
    \varepsilon_1 a_{j01} & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    \varepsilon_k a_{j0k} & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[(k+1) \times (k+1)\]  \hspace{1cm} \text{(1.16)}

The Riemann curvature of the 2-dimensional section of \(M\) spanned by the vectors \((e_i)|_P, \ 1 \leq i \leq k\) and \((e_0)|_P\) can be given by

\[
K(e_i, e_0) = -\varepsilon_i < \overline{D} e_i e_0, D e_i e_0 > = \sum_{j=1}^{n-k-1} \varepsilon_i (a_{0i}^j)^2. \hspace{1cm} \text{(1.17)}
\]

The mean curvature of \(M\) is

\[
H = \frac{1}{k+1} V(e_0, e_0). \hspace{1cm} \text{(1.18)}
\]

\[2. \text{ On the Minimality and Total Developability of the} \]
\[\text{Time-Like Ruled Surfaces with the Time-Like Generating} \]
\[\text{Space in the Minkowski Space} \]

**Theorem 1.** Let \(M\) be a \((k+1)\)-dimensional time-like ruled surface and \(\{e_1, e_2, \ldots, e_k\}\) be an orthonormal base field of the time-like generating space \(E_k(s)\). The lines, which correspond to \(\{e_1, e_2, \ldots, e_k\}\) are asymptotics and geodesics of \(M\).

**Proof.** Since the lines, corresponding to the orthonormal base field vectors \(\{e_1, e_2, \ldots, e_k\}\) of the time-like generating space \(E_k(s)\) are geodesics of \(IR^n_1\), we have

\[\overline{D} e_i e_i = 0, \ 1 \leq i \leq k.\]

From (1.1) we have

\[D e_i e_i = -V(e_i, e_i).\]

Since \(D e_i e_i \in \chi(M)\) and \(V(e_i, e_i) \in \chi^+(M)\), we find

\[D e_i e_i = 0, \quad V(e_i, e_i) = 0.\]
Therefore the lines, corresponding to \( \{ e_1, e_2, ..., e_k \} \) are asymptotics and geodesics of \( M \).

**Theorem 2.** \( M \) is total developable iff \( D_{e_i}e_0 = 0, \ 1 \leq i \leq k \).

**Proof.** Let \( \{ e_0, e_1, ..., e_k \} \) be an orthonormal basis of \( M \) and \( M \) be total developable. Since the system \( \{ e_0, e_1, ..., e_k \} \) is linearly independent, \( D_{e_i}e_0 \) has no component in the normal bundle \( \chi^+(M) \), that is \( V(e_i, e_0) = 0 \).

We know that \( D_{e_i}e_0 = V(e_0, e_i) \). \( (2.1) \)

Since \( V \) is symmetric, from (2.1) we have \( D_{e_i}e_0 = 0 \).

Conversely, assume that \( D_{e_i}e_0 = 0, \ 1 \leq i \leq k \). From the Gauss equation and (2.1) we have \( V(e_i, e_0) = 0 \). If we set this in the Gauss equation, we find \( D_{e_0}e_i = D_{e_0}e_i \).

Therefore, \( D_{e_0}e_i \in sp\{ e_0, e_1, ..., e_k \} \)

Thus we observe that \( \text{rank} [e_0, e_1, ..., e_k, D_{e_0}e_1, D_{e_0}e_2, ..., D_{e_0}e_k] = k + 1 \). \( \square \)

**Theorem 3.** \( M \) is total developable and minimal iff \( M \) is totally geodesic.

**Proof.** If \( X, Y \in \chi(M) \), we have \( X = \sum_{i=1}^{k} a_i e_i + ae_0, \ Y = \sum_{j=1}^{k} b_j e_j + be_0 \).

Therefore we find \( V(X, Y) = \sum_{i=1}^{k} (a_i b_j + b_j a_i) V(e_0, e_i) + ab V(e_0, e_0) + \sum_{i,j=1}^{k} a_i b_j V(e_i, e_j) \).
Since \( V(e_i, e_j) = 0 \) and \( M \) is minimal and total developable we have
\[
V(X, Y) = 0 \quad \text{for all} \quad X, Y \in \chi(M).
\]

Conversely, let \( V(X, Y) = 0 \), for all \( X, Y \in \chi(M) \). Then we have the following relations:
\[
V(e_0, e_i) = 0, \quad V(e_0, e_0) = 0, \quad \text{and} \quad V(e_i, e_j) = 0, \quad 1 \leq i, j \leq k.
\]

By using these equations and (2.1) we find \( \overline{D}_e e_0 = 0, \quad 1 \leq i \leq k \), and so, \( M \) is total developable. Moreover, \( V(e_0, e_0) = 0 \) implies that \( H = 0 \). Therefore \( M \) is minimal. \( \square \)

Let \( \{e_0, e_1, ..., e_k\} \) an orthonormal basis of \( \chi(M) \) and \( \{\zeta_1, \zeta_2, ..., \zeta_{n-k-1}\} \) an orthonormal basis of \( \chi^\perp(M) \). Moreover, we can give co-variant derivative equations of the orthonormal basis \( \{e_0, e_1, ..., e_k, \zeta_1, ..., \zeta_{n-k-1}\} \) of \( \chi(IR^n_1) \), as follows:
\[
\overline{D}_e e_r = \sum_{i=0}^{k} c_r^i e_i + \sum_{m=1}^{n-k-1} c_r(k+m) \zeta_m, \quad 0 \leq r \leq k,
\]
\[
\overline{D}_e \zeta_j = \sum_{i=0}^{k} c_{(k+j)t} e_i + \sum_{m=1}^{n-k-1} c_{(k+j)(k+m)} \zeta_m, \quad 1 \leq j \leq n - k - 1.
\] (2.2)

If we calculate the coefficient \( c_{st} \), \( 0 \leq s, t \leq n - 1 \), and write the equation (2.2) in the matrix form we obtain:
\[
\begin{bmatrix}
\overline{D}_e e_0 \\
\overline{D}_e e_1 \\
\vdots \\
\overline{D}_e e_k \\
\overline{D}_e \zeta_1 \\
\vdots \\
\overline{D}_e \zeta_{n-k-1}
\end{bmatrix}
\]
By using the equation (2.3) we can give the following theorem.

**Theorem 4.** Let $M$ be a $(k+1)$-dimensional time-like ruled surface in $IR^n_1$, $\{e_1, e_2, \ldots, e_k\}$ be an orthonormal base field of the time-like generating space $E_k(s)$ and let the base curve $\alpha(s)$ be an orthonormal trajectory of $E_k(s)$. Then the following propositions are equivalent:

(i) $M$ is total developable,

(ii) The Riemannian curvature $K(e_i, e_0)$ of $M$ is zero, $1 \leq i \leq k$,

(iii) In the equation (2.3) $c_{rs} = 0$, $1 \leq r \leq k$, $k + 1 \leq s \leq n - 1$,

(iv) $A_{\zeta_j}(e_i) = 0$, $1 \leq i \leq k$, $1 \leq j \leq n - k - 1$,

(v) $\mathbf{D}e_0 e_i \in \chi(M)$.

**Proof.** (i) $\Rightarrow$ (ii): We assume that $M$ is total developable. Then by the Theorem 2 and the equation (1.17) we find $K(e_i, e_0) = 0$, $1 \leq i \leq k$.

(ii) $\Rightarrow$ (iii): Let $K(e_i, e_0) = 0$. From (1.15) and (1.17) we find

$$< \mathbf{D}e_0 \zeta_j, e_i > = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n - k - 1.$$
This equation shows that $\overline{D}_{e_0} \zeta_j$ has no component in the directions of $\{e_1, e_2, ..., e_k\}$. Hence we have

$$c_{rs} = 0, \quad 1 \leq r \leq k, \quad k + 1 \leq s \leq n - 1,$$

in the equation (2.3).

(iii) $\Rightarrow$ (iv): Now, we assume that $c_{rs} = 0$. Then from (2.2) we obtain

$$< \overline{D}_{e_0} \zeta_j, e_i > = - \epsilon_i c_{is} = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n - k - 1.$$

Therefore, from (1.17),

$$< \overline{D}_{e_0} \zeta_j, e_i > = \epsilon_i a^j_{0i},$$

then

$$a^j_{0i} = 0.$$

We know that from (1.17)

$$< \overline{D}_e \zeta_j, e_r >= 0.$$

Then, from last two equations, we obtain

$$A_{\zeta_j}(e_i) = 0.$$

(iv) $\Rightarrow$ (v): Let $A_{\zeta_r}(e_i) = 0$. Then from (1.17) we have

$$a^j_{0i} = 0,$$

and $\overline{D}_{e_0} \zeta_j$ has no component in the directions of $\{e_1, e_2, ..., e_k\}$, i.e.

$$c_{rs} = 0, \quad 1 \leq r \leq k, \quad k + 1 \leq s \leq n - 1.$$

Then from (2.3) we have

$$< \overline{D}_{e_0} \zeta_j, e_i >= 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n - k - 1.$$

Since

$$< \overline{D}_{e_0} \zeta_j, e_i > = - < \overline{D}_{e_0} e_i, \zeta_j > = 0, \quad \text{then } \overline{D}_{e_0} e_i \in \chi(M).$$
(v) ⇒ (i): Let $\overline{D}_{e_0}e_i \in \chi(M)$. This means that
\[
\overline{D}_{e_0}e_i \in \text{sp}\{e_0, e_1, ..., e_k\}.
\]
Therefore,
\[
\text{rank} [e_0, e_1, ..., e_k, \overline{D}_{e_0}e_1, ..., \overline{D}_{e_0}e_k] = k + 1.
\]
This means that $M$ is total developable.

References
