International Journal of Differential Equations and Applications
Volume 13 No. 1 2014, 1-15
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v13i1.1364

acadpubl.eu

EXISTENCE OF SOLUTIONS FOR ANISOTROPIC DISCRETE BOUNDARY VALUE PROBLEMS OF KIRCHHOFF TYPE

Zehra Yucedag

Department of Mathematics
Faculty of Science
Dicle University
Diyarbakir, 21280, TURKEY

Abstract: In the present paper, by using variational approach and applying the Mountain Pass theorem, we obtain the existence of at least one nontrivial solution for an anisotropic discrete boundary value problem of $p(k)$-Kirchhoff type in T-dimensional Hilbert space.

AMS Subject Classification: 47A75, 35B38, 35P30, 34L05, 34L30, 35A15
Key Words: Kirchhoff type equation, discrete boundary value problem, varitional methods, critical point, mountain pass theorem

1. Introduction

Kirchhoff [14] established a model, which is called Kirchhoff equation, given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's

Received: November 13, 2013

© 2014 Academic Publications, Ltd.
url: www.acadpubl.eu

wave equation, by considering the effects of the changes in the length of the strings during the vibrations. A distinguish feature of the Kirchhoff equation (1.1) is that the equation contains a nonlocal coefficient $\frac{P_0}{h}+\frac{E}{2L}\int\limits_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$ which depends on the average $\frac{E}{2L}\int\limits_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$ of the kinetic energy $\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2$ on $[0,L]$, and hence the equation is no longer a pointwise identity.

In the present paper, we are concerned with the following problem

$$\begin{cases} -M\left(\frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)}\right)\Delta\left(|\triangle u\,(k-1)|^{p(k-1)-2}\triangle u\,(k-1)\right) \\ \qquad\qquad = f\,(k,u(k))\,, \quad k\in\mathbb{Z}\,[1,T]\,, \\ u\,(0)=u\,(T+1)=0, \end{cases} \tag{P}$$

where $T\geq 2$ is a positive integer, $\mathbb{Z}\,[a,b]$ denotes the discrete interval $\{a,a+1,...,b\}$ with a and b are integers such that $a<b$ is a positive constant, $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator and $f:\mathbb{Z}\,[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Moreover, we define the function $p:\mathbb{Z}\,[0,T]\to[2,\infty)$ such that

$$p^-=\min_{k\in\mathbb{Z}[0,T]}p\,(k)\leq p^+=\max_{k\in\mathbb{Z}[0,T]}p\,(k)\,.$$

In the rest of the present paper, we always assume that the function $M:(0,+\infty)\to(0,+\infty)$ is continuous and nondecreasing.

It is well known that in different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modelling of important questions leads naturally to the consideration of nonlinear difference equations. For this reason, in recent years, many authors have widely developed various methods and techniques, such as fixed points theorems or upper and lower solutions methods, to study discrete problems. Here, we are interested in investigating nonlinear discrete boundary value problems by using variational approach. We refer to [2 − 7, 10, 13, 15, 16, 17, 23, 24] and the reference therein.

We note that problem (**P**) is the discrete variant of the variable exponent anisotropic problem

$$\begin{cases} -\sum\limits_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)=f\,(x)\,, \text{ for } x\in\Omega \\ u=0, \text{ for } x\in\partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary,$f \in L^{\infty}(\Omega)$, and $p_i(x)$ ise continuous on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$ and all $i \in \mathbb{Z}[1, N]$ [19, 20]. Problems like (1.2) have been intensively studied in the last decades, since they can model various phenomena arising from the study of elastic mechanics [25], electrorheological fluids [1, 9, 12, 18, 21] and image restoration [8]. In [8], Chen et al. study a functional with variable exponent $1 \leq p(x) \leq 2$ which provides a model for image denoizing, enhancement and restoration.

In [17], the authors studied the following problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\,\Delta u(k-1)\right) = \lambda\,|u(k)|^{q(k)-2}\,u(k), \\ u(0) = u(T+1) = 0, \qquad k \in \mathbb{Z}[1, T], \end{cases} \tag{1.3}$$

where $T \geq 2$ is a positive integer, the functions $p : \mathbb{Z}[0, T] \to [2, \infty)$ and $q : \mathbb{Z}[1, T] \to [2, \infty)$ are bounded and λ is a positive constant. By using the Mountain Pass theorem and Ekeland's variational principle, they showed the existence of a continuous spectrum of eigenvalues for the problem (1.3).

In [15], the authors dealt with the following problem which is a generalization of (1.3)

$$\begin{cases} -\Delta\left(a(k-1, \Delta u(k-1))\right) = f(k), \qquad k \in \mathbb{Z}[1, T] \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.4}$$

where $T \geq 2$ is a positive integer, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator and $a(k, \xi) = |\xi|^{p(k)-2}\,\xi$ such that $\forall k \in \mathbb{Z}[0, T]$ and $\xi \in \mathbb{R}$. The authors used a minimization method in order to establish some existence results of problem (1.4).

In [16], the authors studied the following problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\,\Delta u(k-1)\right) + |u(k-1)|^{p(k-1)-2}\,u(k-1) \\ \qquad\qquad = \lambda f(k, u(k)), \\ u(0) = u(T+1) = 0, \qquad k \in \mathbb{Z}[1, T], \end{cases} \tag{1.5}$$

where $f(k, t) = |t|^{q(k)-2}\,t - |t|^{s(k)-2}\,t$ for all $t \in \mathbb{R}^+$, $T \geq 2$ is a positive integer, λ is a positive constant and $p : \mathbb{Z}[0, T] \to [2, \infty)$ and $q, s : \mathbb{Z}[1, T] \to [2, \infty)$ are bounded functions. By using the Ricceri's three critical points theorem and variational method, the authors proved the existence and multiplicity of solutions for problem (1.5).

In [11], the authors investigated the following problem

$$\begin{cases} \Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\,\Delta u(k-1)\right) = \lambda f\left(k, u(k)\right), \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$ is a numerical parameter, $f : [0, T+1] \times \mathbb{R}^{T+2} \to \mathbb{R}$, $[a, b]$ for $a < b$, $a, b \in \mathbb{Z}$ denotes a discrete interval $\{a, a+1, ..., b\}$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator; $p : [0, T+1] \to \mathbb{R}_+$, $p^- = \min\limits_{k\in[0,T+1]} p(k) > 1$ and $p^+ = \max\limits_{k\in[0,T+1]} p(k) > 1$; $q : [0, T+1] \to \mathbb{R}_+$ stands for the conjugate exponent. By using the mountain pass technique and a modified version of a finite dimensional three critical point theorem, the authors proved the existence of at least one nontrivial solution and at least two nontrivial solutions for problem (1.6).

For the case $p = const$, In [6] the authors investigated the following problem

$$\begin{cases} -\Delta\left(\phi_p\left(\Delta u(k-1)\right)\right) = \lambda f\left(k, u(k)\right), \quad k \in \mathbb{Z}\left[1, T\right] \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.7}$$

where $T \geq 2$ is a positive integer, $[1, T]$ is the discrete interval $\{1, ..., T\}$, λ is a positive real parameter, $\phi_p(s) = |s|^{p-2}\, s$, $1 < p < \infty$ and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By using the Ricceri's three critical points theorem, they showed the existence of multiple solutions for problem (1.7).

For the case $p = const$, In [13] the authors investigated the following problem

$$\begin{cases} -\Delta_p x(k-1) = f\left(k, x(k)\right), \quad \forall k \in \mathbb{Z}\left[1, T\right] := \{1, 2, ..., T\}, \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \end{cases} \tag{1.8}$$

where Δ_p is the discrete $p-$Laplace operator, i.e., $\Delta_p x(k-1) := \varphi_p\left(\Delta x(k)\right) - \varphi_p\left(\Delta x(k-1)\right)$, $\varphi_p : \mathbb{R} \to \mathbb{R}$ is $\varphi_p(x) = |x|^{p-2}\, x, \forall x \in \mathbb{R}$, $1 < p < \infty$ and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.The authors used critical points theorem to obtain the existence of ground state solutions for problem (1.8).

In the present paper, by help of the Mountain-Pass theorem, we obtain the existence at least one nontrivial solution of problem (**P**). Moreover, as far as we know, there is no any paper which deals with anisotropic discrete boundary value problems of $p\,(k)$-Kirchhoff type.

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, using the variational method we obtain some

existence results for (**P**). In Section 4, by help of the results established in Section 3 we present some corollaries.

2. Preliminaries

Let define the function space,

$$W = \{u : \mathbb{Z}[0, 1+T] \to \mathbb{R};\ \text{such that}\ \ u(0) = u(T+1) = 0\}.$$

Then, W is a T-dimensional Hilbert space [2] with the inner product

$$\langle u, v\rangle = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1),\ \forall u, v \in W\ .$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

On the other hand, it is useful to introduce other norms on W, namely

$$|u|_m = \left(\sum_{k=1}^{T} |u(k)|^m\right)^{1/m},\ \forall u \in W \text{ and } m \geq 2.$$

It can be verified [7] that

$$T^{2-m/(2m)} |u|_2 \leq |u|_m \leq T^{1/m} |u|_2,\ \forall u \in W \text{ and } m \geq 2. \tag{2.1}$$

Lemma 2.1. (*i*) *Let $u \in W$ and $\|u\| > 1$.Then*

$$\sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} \geq \frac{1}{p^+ \left(\sqrt{T}\right)^{p^- - 2}} \|u\|^{p^-} - T.$$

(*ii*) *Let $u \in W$ and $\|u\| < 1$.Then*

$$\sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} \geq \frac{1}{p^+ \left(\sqrt{T}\right)^{2-p^+}} \|u\|^{p^+}.$$

(*iii*) *For any* $m \geq 2$ *there exist a positive constant* c_m *such that*

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m , \ \forall u \in W .$$

Since the proof of Lemma 2.1 is very similar to the proof of Lemma 1 in [17], we omit it.

Moreover, from (2.1) and Lemma 2.1 (*iii*), it reads

$$|u|_m^m \leq T |u|_2^m \leq c_m T \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{\frac{m}{2}} = c_m T \|u\|^m . \tag{2.2}$$

3. The Main Results

The energy functional corresponding to problem (**P**) is defined as $I : W \to \mathbb{R}$,

$$I(u) = \widehat{M} \left(\sum_{k=1}^{T+1} \frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)} \right) - \sum_{k=1}^{T} F(k, u(k)) ,$$

where $k \in \mathbb{Z}[1,T]$, $\widehat{M}(t) = \int_0^t M(s)\, ds$ and $F(k,t) = \int_0^t f(k,\xi)\, d\xi$. The functional I is differentiable in the sense of Gâteaux, and its Gâteaux derivative is

$$\begin{aligned} \left\langle I'(u), v \right\rangle =& M \left(\sum_{k=1}^{T+1} \frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)} \right) \\ & \times \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)-2} \triangle u(k-1) \triangle v(k-1) \\ & - \sum_{k=1}^{T} f(k, u(k)) v(k) , \end{aligned}$$

for all $u, v \in W$.

Definition 3.1. We say that $u \in W$ is a weak solution of (**P**) if

$$M\left(\sum_{k=1}^{T+1}\frac{|\triangle u\,(k-1)|^{p(k-1)}}{p(k-1)}\right)\sum_{k=1}^{T+1}|\triangle u\,(k-1)|^{p(k-1)-2}\triangle u\,(k-1)\triangle\varphi\,(k-1)$$
$$=\sum_{k=1}^{T}f\,(k,u\,(k))\,\varphi\,(k)\,,\quad k\in\mathbb{Z}\,[1,T]\,,$$

for any $\varphi\in W$.

Since we work in finite dimensional space, summing by parts we see that any weak solution of (**P**) is in fact a strong one (see [**11**]). Hence in order to solve (**P**) we need to find critical points to I .

Lemma 3.2. (i) *The functional I is well defined on W.*

(ii) *The functional I is of class $C^1(W,\mathbb{R})$ and*

$$\left\langle I^{'}(u),v\right\rangle = M\left(\sum_{k=1}^{T+1}\frac{|\triangle u\,(k-1)|^{p(k-1)}}{p(k-1)}\right)$$
$$\times\sum_{k=1}^{T+1}|\triangle u\,(k-1)|^{p(k-1)-2}\triangle u\,(k-1)\triangle v\,(k-1)$$
$$-\sum_{k=1}^{T}f\,(k,u\,(k))\,v\,(k)\,,$$

for all $u,v\in W$.

Since the statements $(i)-(ii)$ can be obtained by standard arguments, we omit the proof of Lemma 3.2.

Theorem 3.3. *Assume the following conditions hold:*

(M_1) *:(Polynomial growth condition) There exist positive real numbers A,B and $A\leq B$ and $\alpha\geq 1$ such that*

$$As^{\alpha-1}\leq M(s)\leq Bs^{\alpha-1}\text{ for }s\geq s_*>0;$$

$(f_0):f:\mathbb{Z}\,[1,T]\times\mathbb{R}\to\mathbb{R}$ *satisfies condition*

$$|f(k,t)|\leq c_0\left(1+|t|^{\gamma(k)-1}\right),$$

*where $1\leq\gamma^+<\alpha p^-$ and $c_0>0$.Then, the problem (**P**) has at least one solution*

Proof. Let $\|u\| > 1$. By (M_1), (f_0) and from in (i) Lemma 2.1

$$I(u) \geq A \int_0^{\sum_{k=1}^{T+1} \frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)}} \xi^{\alpha-1} d\xi - c \sum_{k=1}^{T} \frac{|u(k)|^{\gamma(k)}}{\gamma(k)} - \widehat{c_0}.$$

If we use the inequalities $|u(k)|^{\gamma(k)} \leq |u(k)|^{\gamma^-} + |u(k)|^{\gamma^+}$, for $k \in \mathbb{Z}[1,T]$ and (2.2) in the above inequality, we write

$$\begin{aligned} I(u) \geq & \frac{A}{\alpha} \left(\sum_{k=1}^{T+1} \frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)} \right)^{\alpha} \\ & - \frac{c}{\gamma^-} \left(\sum_{k=1}^{T} |u(k)|^{\gamma^+} + \sum_{k=1}^{T} |u(k)|^{\gamma^-} \right) - \widehat{c_0} \\ \geq & \frac{A}{\alpha \left(p^+ \left(\sqrt{T} \right)^{p^- - 2} \right)^{\alpha}} \|u\|^{\alpha p^-} - K(\alpha, T) T^{\alpha} \\ & - \frac{c}{\gamma^-} \left(c_{\gamma^+} T \|u\|^{\gamma^+} + c_{\gamma^-} T \|u\|^{\gamma^-} \right) - \widehat{c_0} \\ \geq & \frac{A}{\alpha \left(p^+ \left(\sqrt{T} \right)^{p^- - 2} \right)^{\alpha}} \|u\|^{\alpha p^-} - K(\alpha, T) T^{\alpha} \\ & - \frac{c_1}{\gamma^-} T \|u\|^{\gamma^+} - \widehat{c_0} \to +\infty \text{ as } \|u\| \to +\infty \end{aligned}$$

where $K(\alpha, T) > 0$. As $\alpha p^- > \gamma^+ \geq 1$, then I is coercive.

A continuous coercive functional on a finite dimensional space is bounded from below. On the other hand, it is obvious that it is also weakly lower semicontinuous on the finite dimensional Hilbert space W. Therefore, $(\mathbf{P})$ has at least one solution in W. The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Assume (M_1) and the following conditions hold:*

$(f_1) : f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|f(k,t)| \leq c \left(1 + |t|^{\beta(k)-1} \right), \ |t| \geq t_*, \ \forall k \in \mathbb{Z}[1,T]$$

where $\beta^- > \alpha p^+$ and $c > 0$;

$(f_2) : f(k,t) = o(|t|^{\alpha p^+ - 1})$, $t \to 0$, *for* $\forall k \in \mathbb{Z}[1,T]$;

(AR) : *Ambrosetti-Rabinovitz's condition holds, i.e.* $\theta > \frac{B}{A}\alpha p^+$ *such that*

$$0 \leq \theta F(k,t) \leq f(k,t)t, \quad |t| \geq t_*, \ \forall k \in \mathbb{Z}[1,T];$$

(f_3) : *There exists* $t_* > 0$ *such that* $F(k,t) > 0$ *for* $\forall k \in \mathbb{Z}[1,T]$ *and all* $|t| \geq t_*$;

Then, the problem $(\mathbf{P})$ *has at least one nontrivial solution.*

Definition 3.5. We say that I satisfies Palais-Smale condition ((PS) condition for short) if every sequence $\{u_n\} \subset W$ such that $|I(u_n)| \leq c$ and $I'(u_n) \to 0$ contains a convergent subsequence in the norm of W.

Lemma 3.6. *Suppose* (M_1), (f_1) *and* (AR) *hold. Then* I *satisfies* (PS) *condition.*

Proof. Assume that $\{u_n(k)\} \subset W$, $|I(u_n(k))| \leq c$ and $I'(u_n(k)) \to 0$ as $n \to \infty$. Then by (M_1), we have

$$\begin{aligned} c + \|u_n\| \geq & I(u_n) - \frac{1}{\theta}\left\langle I'(u_n), u_n \right\rangle \\ = & \widehat{M}\left(\sum_{k=1}^{T+1} \frac{|\triangle u_n(k-1)|^{p(k-1)}}{p(k-1)}\right) - \sum_{k=1}^{T} F(k, u_n(k)) \\ & - \frac{1}{\theta} M\left(\sum_{k=1}^{T+1} \frac{|\triangle u_n(k-1)|^{p(k-1)}}{p(k-1)}\right) \sum_{k=1}^{T+1} |\triangle u_n(k-1)|^{p(k-1)} \\ & + \frac{1}{\theta}\sum_{k=1}^{T} f(k, u_n(k)) u_n(k) \\ \geq & \left(\frac{A}{\alpha} - \frac{Bp^+}{\theta}\right)\left(\sum_{k=1}^{T+1} \frac{|\triangle u_n(k-1)|^{p(k-1)}}{p(k-1)}\right)^{\alpha} - \widehat{c}. \end{aligned}$$

By (AR) and Lemma 2.1 (i), we can write

$$c + \|u_n\| \geq \left(\frac{A}{\alpha} - \frac{Bp^+}{\theta}\right) \frac{1}{\left(p^+\left(\sqrt{T}\right)^{p^- - 2}\right)^{\alpha}} \|u\|^{\alpha p^-} - K(\alpha, T) T^{\alpha}.$$

we conclude that $\{u_n\}$ is bounded in W. That information combined with the fact that W is a finite dimensional Hilbert space implies that there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in W$ such that $\{u_n\}$ converges to u_0 in W. Thus, I satisfies (PS) condition. □

Remark 3.6. Let $\alpha = 1$.Then $A \leq M(s) \leq B$. If we choose $M(s) = 1$, it reads

$$-\Delta\left(\left|\Delta u(k-1)\right|^{p(k-1)-2}\Delta u(k-1)\right) = f(k,u).$$

Therefore, (AR) holds, i.e. $\theta > p^+$.

Lemma 3.7. (i) *There exist two positive real numbers* ρ *and* a *such that* $I(u) \geq a > 0$, $u \in W$ *with* $\|u\| = \rho$.

(ii) *There exists* $u \in W$ *such that* $\|u\| > \rho$, $I(u) < 0$.

Proof. (i) Let $\|u\| < 1$. Then by (M_1) and Lemma 2.1 (ii), we have

$$I(u) \geq \frac{A}{\alpha\left(p^+\left(\sqrt{T}\right)^{2-p^+}\right)^{\alpha}}\|u\|^{\alpha p^+} - \sum_{k=1}^{T}F(k,u(k)) - \widetilde{c}.$$

Let $\varepsilon > 0$ be small enough such that $\varepsilon c_{\alpha p^+}T \leq \frac{A}{2\alpha}\cdot\frac{1}{\left(p^+\left(\sqrt{T}\right)^{2-p^+}\right)^{\alpha}}$. By (f_1) and (f_2), we get $F(k,t) \leq \varepsilon|t|^{\alpha p^+} + c|t|^{\beta(k)}$, $c > 0$, $t \in \mathbb{R}$. Therefore, by (2.2) it follows

$$\begin{aligned}
I(u) \geq & \frac{A}{\alpha\left(p^+\left(\sqrt{T}\right)^{2-p^+}\right)^{\alpha}}\|u\|^{\alpha p^+} - \varepsilon\sum_{k=1}^{T}|u(k)|^{\alpha p^+} - c\sum_{k=1}^{T}|u(k)|^{\beta(k)} - \widetilde{c} \\
\geq & \frac{A}{\alpha\left(p^+\left(\sqrt{T}\right)^{2-p^+}\right)^{\alpha}}\|u\|^{\alpha p^+} - \varepsilon c_{\alpha p^+}T\|u\|^{\alpha p^+} - cc_{\beta^+}T\|u\|^{\beta^+} \\
& - cc_{\beta^-}T\|u\|^{\beta^-} - \widetilde{c} \\
\geq & \frac{A}{2\alpha\left(p^+\left(\sqrt{T}\right)^{2-p^+}\right)^{\alpha}}\|u\|^{\alpha p^+} - 2\max\left\{cc_{\beta^+}T, cc_{\beta^-}T\right\}\|u\|^{\beta^-} - \widetilde{c}.
\end{aligned}$$

Since$\|u\| < 1$ and $\alpha p^+ < \beta^-$, there exist two positive real numbers ρ and a such that $I(u) \geq a > 0$, $u \in W$ with $\|u\| = \rho \in (0,1)$.

(ii) From (AR) and (f_3), one easily deduces

$$F(k,t) \geq \frac{F(k,t_*)}{t_*^{\theta}}t^{\theta},\ |t| \geq t_*$$

so that $\forall\delta > 1$ and $\forall u \in W$ nonnegative such that the sets

$$E_1 := \{k \in \mathbb{Z}[1,T] : u(k) \geq t_*\}$$

and

$$E_2 := \{k \in \mathbb{Z}[1,T] : \delta u(k) \geq t_*\},$$

we get

$$\begin{aligned}\sum_{k=1}^{T} F(k,\delta u(k)) &\geq \sum_{E_2} F(k,\delta u(k)) \geq \frac{\delta^\theta}{t_*^\theta}\sum_{E_2} F(k,t_*)u^\theta(k) \\ &\geq \frac{\delta^\theta}{t_*^\theta}\sum_{E_1} F(k,t_*)u^\theta(k) \geq \delta^\theta \sum_{E_1} F(k,t_*) > 0,\end{aligned}$$

(recall that $F > 0$ and $F(.,t_*) > 0$ almost everywhere).

When $|t| \geq t_*$, from (M_1) we can easily obtain that (since $\frac{B}{A} \geq 1$) $\widehat{M}(t) \leq \frac{B}{\alpha}t^\alpha \leq \frac{B}{\alpha}t^{\frac{B}{A}\alpha}$. For $\omega \in W \setminus \{0\}$, we have

$$\begin{aligned}I(t\omega) &= \widehat{M}\left(\sum_{k=1}^{T+1}\frac{|t\triangle\omega(k-1)|^{p(k-1)}}{p(k-1)}\right) - \sum_{k=1}^{T} F(k,t\omega(k)) - \widetilde{c} \\ &\leq \frac{B}{\alpha\left(p^-\right)^{\frac{B}{A}\alpha}} t^{\frac{B}{A}\alpha p^+}\sum_{k=1}^{T+1}|\triangle\omega(k-1)|^{p(k-1)} - t^\theta\sum_{E_1}F(k,\omega) - \widetilde{c}.\end{aligned}$$

From the assumption on θ (see (AR)), so this implies $I(t\omega) \to -\infty$ as $t \to +\infty$. The proof of Theorem 3.4 is complete.

Proof of Theorem 3.4, from Lemma 3.6, Lemma 3.7 and the fact that $I(0) = 0$, I satisfies the Mountain Pass theorem [**22**]. Therefore, (**P**) has at least one nontrivial solution. □

Remark 3.8. When W is a finite dimensional Banach space, the condition

$$(iii) \lim_{\|u\|\to\infty} I(u) = -\infty$$

implies (PS) condition, i.e. Lemma 3.6 and Lemma 3.7 (ii).

Moreover, from the assumption $\beta^- > \alpha p^+$ given in (f_1) Theorem 3.4, it is obvious that the functional I is anti-coercive. Therefore, there may no any necessity to prove Lemma 3.6 and Lemma 3.7 (ii).

Example 3.9. In promlem (**P**), if we choose $T = 2$, $\alpha = 2$, $A = 2$, $B = 4$, $k = 1, 2$ and p, β, f as follows

$$\beta(k) = 17k + 19, \quad p(k-1) = k + 2, \quad f(k,u) = |u|^{17k+18}$$

then we have

$$\begin{aligned} p^{-} &= 3,\ p^{+}=4,\ \theta>16, \beta^{-}=36,\ \beta^{+}=53, \\ F(k,u) &= \frac{1}{17k+19}\ |u|^{17k+19}. \end{aligned}$$

Therefore, the functions p, β, f and F satify the conditions given in Theorem 3.4.

4. Corollaries

Now, we present some consequences of Theorem 3.3 and Theorem 3.4. Let consider the following version of the discrete Kirchhoff type equation

$$-\left(a+b\left(\frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)}\right)^{\alpha-1}\right)\Delta\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right) = f(k,u(k)), \quad k\in\mathbb{Z}[1,T], \quad (4.1)$$

where $\alpha\geq 1$ and a,b are two positive constants. We must point out that when $\alpha=2$, equation (4.1) turns to the discrete Kirchhoff type equation

$$-\left(a+b\left(\frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)}\right)\right)\Delta\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right) = f(k,u(k)), \quad k\in\mathbb{Z}[1,T].$$

Let $M(t)=a+bt^{\alpha-1}$ with $t=\frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)}$. Then the condition (M_1) holds: Let choose $A\leq b<B$. It is clear $At^{\alpha-1}\leq a+bt^{\alpha-1}\leq Bt^{\alpha-1}$ for all $t\geq t^{*}$. Therefore, corresponding to Theorem 3.3 and Theorem 3.4, we have the following corollaries.

Corollary 4.1. *If (M_1) and (f_0) hold, then (4.1) has at least one solution.*

Corollary 4.2. *If (M_1) and $(f_1),(f_2)$ and (f_3) hold, then (4.1) has at least one nontrivial solution.*

Acknowledgments

The author would like to thank Prof. Dr. Rabil Ayazoglu (Mashiyev) for his generous advice and support.

References

[1] E. Acerbi, G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew Math.*, **258** (1992), 761-766.

[2] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Anal.*, **58** (2004), 69-73.

[3] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, *Adv. Diff. Equ.*, **2** (2005), 93-99.

[4] G. Bonanno, P. Candito, Nonlinear difference equations investigated via critical point methods, *Nonlinear Ana*, **70** (2009), 3180-3186.

[5] A. Cabada, A. Lannizzotto, S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.*, **356** (2009), 418-428.

[6] P. Candito, N. Giovannelli, Multiple solutions for a discrete boundary value problem involving the p-Laplacian, *Math. Appl. Comput.*, **56** (2008), 959-964.

[7] X. Cai, J. Yu, Existence theorems for second-order discrete boundary value problems, *J. Math. Anal. Appl.*, **320** (2006), 649-661.

[8] Y. Chen, S. Levine, M. Rao, Variable exponent linear growth functionals in image processing, *SIAM J. Appl. Math.*, **66**, No. 4 (2006), 1383-1406.

[9] L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, PhD Thesis, University of Frieburg, Germany, 2002.

[10] A. Guiro, I. Nyanquini, S. Ouaro, On the solvability of discrete nonlinear Neumann problems involving the $p(x)$-Laplacian, *Adv. Diff. Equ.*, **32** (2011).

[11] M. Galewski, S. Glkab, On the discrete boundary value problem for anisotropic equation, *J. Math. Anal. Appl.*, **386** (2012), 956-965.

[12] T.C. Halsey, Electrorheological fluids, *Science*, **258** (1992), 761-766.

[13] P. Jebelean, C. Şerban, Ground state periodic solutions for difference equations with discrete p-Laplacian, *Appl. Math. Comput.*, **217** (2011), 9820-9827.

[14] G. Kirchhoff, *Mechanic*, Leipzig, Teubner, 1883.

[15] B. Kone, S. Ouaro, Weak solutions for anisotropic discrete boundary value problems, *J. Diff. Equ. Appl.* (2010), 1-11.

[16] R.A. Mashiyev, Z. Yucedag, S. Ogras, Existence and multiplicity of solutions for a Dirichlet problem involving the discrete $p(x)$-Laplacian operator, *E.J. Qualitative Theory of Diff. Equ.*, **67** (2011), 1-10.

[17] M. Mihăilescu, V. Rădulescu, S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems, *J. Difference Equ. Appl.*, **15** (2009), 557-567.

[18] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. A Math. Phys. Eng. Sci.*, **462** (2006), 2625-2641.

[19] M. Mihăilescu, P. Pucci, V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, *C.R. Acad. Sci. Paris Ser I*, **345** 2007, 561-566.

[20] M. Mihăilescu, P. Pucci, V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.*, **340** (2008), 687-698.

[21] M. Růžička, *Electrorheological Fluids. Modeling and Mathematical Theory*, Berlin, Springer-Verlag, 2002.

[22] M. Willem, *Minimax Theorems*, Birkhauser, Basel, 1996.

[23] J. Yu, Z. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, *J. Math. Anal. Appl.*, **231** (2006), 18-31.

[24] G. Zhang, S. Liu, On a class of semipositone discrete boundary value problem, *J. Math. Anal. Appl.*, **325** (2007), 175-182.

[25] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.*, **29** (1987), 33-66.