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AN ORDER SEVEN CONTINUOUS EXPLICIT METHOD FOR DIRECT SOLUTION OF GENERAL FIFTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

S.J. Kayode

Department of Mathematical Sciences
School of Sciences
Federal University of Technology
P.M.B. 704, Akure, 234, Ondo State, NIGERIA

Abstract: This paper provides an explicit continuous method of order seven for direct integration of fifth-order general differential equations without reduction to systems of lower order equations. The explicit method requires no main predictor as a starting value for implementation. In the derivation of the method, collocation and interpolation procedures are adopted using power series as the basis function. The purpose of this work is to produce a high order explicit method that could be used as a predictor for the implementation of implicit methods in predictor-corrector mode in order to enhance the accuracy and efficiency of such methods. The method is symmetric, normalized, consistent and zero-stable. It is suitable for both non-stiff and mildly-stiff problems.

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1. Introduction

In this paper, an explicit linear multistep method of the form

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$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^r \sum_{j=0}^{k-1} \beta_j f_{n+j}, \quad k \geq 6 \tag{1}$$

for direct numerical solution of general fifth-order initial value problems of the form

$$y^{(r)} = f(t,\ y,\ y',\ y'',...,\ y^{(r-1)}),\ \ y^{(r-1)}(t_0) = \mu_{r-1},\ r = 1,...,\ 4,\ 5 \tag{2}$$

on the discrete point-set $\{x_i : x_i = a + ih, i = 0, 1, ...\}$.

Available literature have shown that attention has been directed at solving higher order ordinary differential equations with Linear Multistep Methods [LMMs] without following the conventional method of reduction to systems of first order problems, [see Lambert and Watson, 1976; Awoyemi, 2001; Awoyemi and Kayode, 2002; Pittaluga, Sacripante and Venturino, 2003; Jator, 2007; Kayode, 2008, 2010; Majid, Azani and Suleiman, 2009; Fatokun and Ajibola, 2009; Ehigie, Okunuga, Sofoluwe and Akanbi, 2010; Kayode and Adeyeye, 2011; Akinfenwa, 2013; Kayode and Obarhua, 2013]. This essentially is to avoid the inherent drawbacks associated with such methods and to improve accuracy. The many research activities along this line have produced numerical schemes that are incapable of handling fifth order problems directly without reducing them to lower order problems.

In the work of Kayode and Awoyemi, 2010, an order six multiderivative implicit (closed) method was proposed to solve fifth order ordinary differential equations. In that paper, an explicit method of order six was used as a main predictor for its implementation. In this present work, a higher order explicit (open) method is proposed for direct solution of fifth order ordinary differential equations for the purpose of enhancing accuracy. This new method requires no main stating values as it was in that of Kayode and Awoyemi, 2010.

2. Materials and Method

The approximate solution to problem (2) is taken as a partial sum of a power series of the form

$$y(x) = \sum_{j=0}^{k+2} \lambda_j x^j, \tag{3}$$

Using equation (2) in the fifth derivative of (3) yields

$$\sum_{j=0}^{k+2} j(j-1)(j-2)(j-3)(j-4)\lambda_j x^{j-5} = f(x, y, y', y'', y''', y^{iv}) \tag{4}$$

Collocating equation (3) at $x = x_{n+i},\ i = 1(1)4$ and interpolating (2) at $x = x_{n+j}, j = 0(1)4$ yield the following systems of equations

$$\sum_{j=0}^{k+2} j(j-1)(j-1)(j-2)(j-3)(j-4)\lambda_j x_{n+i}^{j-5} = f_{n+i},\ i = 1(1)4, \tag{5}$$

$$\sum_{j=0}^{k+2} \lambda_j x_{n+j} = y_{n+j}, j = 0(1)4. \tag{6}$$

The system of equations obtained from the collocation and interpolation above is represented by the matrix equation

$$AB = Y, \tag{7}$$

where $A$is an m-square matrix, $B$and $Y$are m-column vectors as defined below

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & \cdots & \gamma x_{n+1}^{k-3} \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+2} & \cdots & \gamma x_{n+3}^{k-3} \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+k-1} & \cdots & \gamma x_{n+5}^{k-3} \\ 1 & x_n & x_n^2 & x_n^3 & & \cdots & & & x_n^{k+2} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & & \cdots & & & x_{n+1}^{k+2} \\ \vdots & & & & \vdots & & & \vdots & \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & x_{n+k-1}^3 & & \cdots & & & x_{n+k-1}^{k+2} \end{bmatrix},$$

$$B = [\lambda_0\ \ \lambda_1\ \ \lambda_2\ ...\ \ \lambda_{k+2}\ \ \lambda_{k+3}]^T,$$

$$Y = [f_{n+1}\ \ f_{n+3}\ \ f_{n+i}\ \ y_n\ \ y_{n+1}\ \ \cdots\ \ y_{n+j}]^T,$$

$$\gamma = j(j-1)(j-2)(j-3)(j-4),\ \ j = 0,\ 1,\ 2,\ ...,\ k+3,$$

$$f_{n+i} = f(x_{n+i},\ y_{n+i},\ y'_{n+i},\ y''_{n+i},\ y'''_{n+i}, y^{iv}_{n+i})$$

and

$$y_{n+i} = y(x_{n+i}),\quad i = 0, 1, 2, \ldots,$$

T is matrix transpose.

Substituting the solution of matrix equation (7) for $\lambda'_j s,\ j = 0, 1, \cdots, k+2$, in the approximate equation (2) produces the continuous method

$$y_k(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j},\ \beta_k = 0 \tag{8}$$

By using the transformation $x = sh + x_{n+5}$, $s \in (0,\ 1]$ and h being steplength, in (8) after necessary simplification, an open method with variable coefficients is obtained as

$$y_k(s) = \sum_{j=0}^{k-1} \alpha_j(s) y_{n+j} + h^5 \sum_{j=0}^{k} \beta_j(s) f_{n+j},\ \beta_k = 0, \tag{9}$$

with the coefficients $\alpha_j(s)$ and $\beta_j(s)$, $j = 0(1)6$, computed to be

$$\begin{aligned}
\alpha_0(s) =& \frac{1}{8400}[1360s + \frac{9964}{3}s^2 + 2856s^3 + \frac{2863}{3}s^4 - \frac{224}{3}s^6 - 16s^7 - s^8],\\
\alpha_1(s) =& \frac{-1}{1680}[940s + \frac{7654}{3}s^2 + 2436s^3 + \frac{2653}{3}s^4 - \frac{224}{3}s^6 - 16s^7 - s^8],\\
\alpha_2(s) =& \frac{1}{840}[240s + \frac{4084}{3}s^2 + 1876s^3 + \frac{2443}{3}s^4 - \frac{224}{3}s^6 - 16s^7 - s^8],\\
\alpha_3(s) =& \frac{1}{840}[1160s + \frac{2006}{3}s^2 - 1176s^3 - \frac{2233}{3}s^4 + \frac{224}{3}s^6 + 16s^7 + s^8],\\
\alpha_4(s) =& \frac{-1}{1680}[5360s + \frac{11876}{3}s^2 - 336s^3 - \frac{2023}{3}s^4 + \frac{224}{3}s^6 + 16s^7 + s^8],\\
\alpha_5(s) =& \frac{1}{8400}[8400 + 16140s + \frac{26786}{3}s^2 + 644s^3 - \frac{1813}{3}s^4 + \frac{224}{3}s^6\\
&+ 16s^7 + s^8],\\
\beta_1(s) =& \frac{1}{40320}[1120s + \frac{8032}{3}s^2 + 2254s^3 + \frac{2233}{3}s^4 - \frac{182}{3}s^6 - 14s^7 - s^8],\\
\beta_3(s) =& \frac{1}{10080}[3160s + \frac{20926}{3}s^2 + 5257s^3 + \frac{4543}{3}s^4 - \frac{266}{3}s^6 - 17s^7 - s^8],\\
\beta_5(s) =& \frac{1}{201600}[4160s + \frac{32456}{3}s^2 + 10934s^3 + \frac{17087}{3}s^4 + 1680s^5\\
&+ \frac{854}{3}s^6 + 26s^7 + s^8.
\end{aligned}$$

Evaluation of equation (9) at the last grid point at where $s = 1$ gives rise to a particular discrete zero-stable open method

$$\begin{aligned}
y_{n+6} = 4y_{n+5} - 5y_{n+4} + 5y_{n+2} - 4y_{n+1} + y_n&\\
+ \frac{h^5}{6}(f_{n+5} + 10f_{n+3} + f_{n+1}),&\quad (10)
\end{aligned}$$

where

$$f_{n+i} = f(x_{n+i},\ y_{n+i},\ y'_{n+i},\ y''_{n+i},\ y'''_{n+i}, y^{iv}_{n+i}), \quad i = 0,\ 1,\ 2,\ \ldots$$

The first, second, third and fourth derivatives of (10) are obtained respectively from (9) as

$$y'_{n+6} = \frac{1}{1260h}(5112y_{n+5} - 7605y_{n+4} - 5160y_{n+3} + 16920y_{n+2} - 12240y_{n+1} + 2973y_n + h^5(573f_{n+5} + 4785f_{n+3} + 492f_{n+1}), \quad (11)$$

$$y''_{n+6} = \frac{1}{6300h^2}(13078y_{n+5} - 2915y_{n+4} - 87620y_{n+3} + 156710y_{n+2} - 13030y_{n+1} + 24197y_n + \frac{h^5}{15120}(149158f_{n+5} + 90352f_{n+3} + 9505f_{n+1}), \quad (12)$$

$$y'''_{n+6} = \frac{1}{600h^3}(144y_{n+5}+1980y_{n+4}-8760y_{n+3}+13440y_{n+2}-8520y_{n+1}+1956y_n) + h^5(1021f_{n+5} + 3175f_{n+3} + 310f_{n+1})), \quad (13)$$

$$y^{(iv)}_{n+6} = \frac{1}{150h^4}(491y_{n+5} - 2305y_{n+4} + 4310y_{n+3} - 3410y_{n+2} + 1555y_{n+1} - 341y_n + \frac{h^5}{24}(8291f_{n+5} - 4120f_{n+3} - 15550f_{n+1}). \quad (14)$$

The open method (10) and its associated derivatives (11), (12), (13) and (14) are tested to be of the same order p = 7. The error constant of (10) is found to be $C_{p+2} \approx -0.125$. According to Lambert (1973) in Awoyemi and Idowu (2005), a consistent and zero stable method satisfies the necessary and sufficient conditions for convergence of LMMs. The derived method (10) was tested to have satisfied these necessary and sufficient conditions for convergence.

3. Implementation of the Method

Implementation of the explicit method (10) does not need a predictor fory_{n+6} and its associated derivatives but only requires starting values for y_{n+i}, $i = 1(1)5$. These starting values and their associated derivatives are given as

$$y_{n+5} = 5y_{n+4} - 10y_{n+3} + 10y_{n+2} - 5y_{n+1} + y_n + \frac{h^5}{24}(5f_{n+4} + 20f_{n+3} - f_n), \quad (15)$$

$$y'_{n+5} = \frac{1}{12h}(77y_{n+4} - 214y_{n+3} + 234y_{n+2} - 122y_{n+1} + 25y_n) + \frac{h^5}{840}(5575f_{n+4} + 18460f_{n+2} - 1014f_n), \quad (16)$$

$$y''_{n+5} = \frac{1}{12h^2}(71y_{n+4} - 236y_{n+3} + 294y_{n+2} - 164y_{n+1} + 35y_n) + \frac{h^5}{120}(1651f_{n+4} + 3996f_{n+2} - 247f_n), \quad (17)$$

$$y'''_{n+5} = \frac{1}{2h^3}(7y_{n+4} - 26y_{n+3} + 36y_{n+2} - 22y_{n+1} + 5y_n) + \frac{h^5}{40}(147f_{n+4} + 206f_{n+2} - 13f_n), \quad (18)$$

$$y^{(iv)}_{n+5} = \frac{1}{h^4}(y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n) + \frac{h^5}{24}(53f_{n+4} + 18f_{n+2} + f_n). \quad (19)$$

Taylor series expansion was used to evaluate $y_{n+i}, y^{(1)}_{n+i}, y^{(2)}_{n+i}, y^{(3)}_{n+i}$, y_{n+i}, $i = 1, ..., k$, $k < 4$. This is due to the fact that the minimum value of k for the development of any non-hybrid Linear Multistep Method (LMM) must be equal to the order of the differential equation it is meant to solve. The series are given as

$$y_{n+i} \equiv y(x_n + ih) \cong y(x_n) + ihy^{(1)}(x_n) + \frac{(ih)^2}{2!}y^{(2)}(x_n) + \frac{(ih)^3}{3!}y^{(3)}(x_n) + \frac{(ih)^4}{4!}y^{(4)}(x_n) + \frac{(ih)^5}{5!}f_n + \cdots,$$

$$y^{(1)}_{n+i} \equiv y^{(1)}(x_n + ih) \cong y^{(1)}(x_n) + ihy^{(2)}(x_n) + \frac{(ih)^2}{2!}y^{(3)}(x_n) + \frac{(ih)^3}{3!}y^{(4)}(x_n) + \frac{(ih)^4}{4!}f_n + \frac{(ih)^5}{5!}f^{(1)}_n + \cdots$$

$$y^{(2)}_{n+i} \equiv y^{(2)}(x_n + ih) \cong y^{(2)}(x_n) + ihy^{(3)}(x_n) + \frac{(ih)^2}{2!}y^{(4)}(x_n) + \frac{(ih)^3}{3!}f_n + \frac{(ih)^4}{4!}f^{(1)}_n + \frac{(ih)^5}{5!}f^{(2)}_n + \cdots$$

$$y_{n+i}^{(3)} \equiv y^{(3)}(x_n + ih) \cong y^{(3)}(x_n) + ihy^{(4)}(x_n) + \frac{(ih)^2}{2!} f_n + \frac{(ih)^3}{3!} f_n^{(1)} + \frac{(ih)^4}{4!} f_n^{(2)} + \cdots$$

$$y_{n+i}^{(4)} \equiv y^{(4)}(x_n + ih) \cong y^{(4)}(x_n) + ihf + \frac{(ih)^2}{2!} f_n^{(1)} + \frac{(ih)^3}{3!} f_n^{(2)} + \frac{(ih)^4}{4!} f_n^{(3)} + \cdots$$

For $m = 5$ in equation (1) and taking $y^{(5)}(x_n) = f_n$, $y_n^{(4+j)} = f_n^{(j)}$, $j = 0, 1, 2, ...,$ $f_n^{(0)} = f_n$ and $f_n^{(j)} = f^{(j)}(x_n, y_n, y_n^{(1)}, y_n^{(2)}, y_n^{(3)}, y^{(4)})$, the values of $f_n^{(1)}$, $f_n^{(2)}$, were obtained by partial derivative technique as

$$f^{(1)} = \frac{df}{dx} = \frac{\partial f}{\partial x} + y^{(1)} \frac{\partial f}{\partial y} + y^{(2)} \frac{\partial f}{\partial y^{(1)}} + y^{(3)} \frac{\partial f}{\partial y^{(2)}} + f \frac{\partial f}{\partial y^{(3)}},$$

$$f^{(2)} = \frac{d^2 f}{dx^2} = 2(Ay^{(1)} + By^{(2)} + Cy^{(3)} + Df + E + F,$$

where

$$A = \frac{\partial^2 f}{\partial x \partial y} + y^{(2)} \frac{\partial^2 f}{\partial y \partial y^{(1)}} + y^{(3)} \frac{\partial^2 f}{\partial y \partial y^{(2)}} + f \frac{\partial^2 f}{\partial y \partial y^{(3)}},$$

$$B = \frac{\partial^2 f}{\partial x \partial y} + y^{(3)} \frac{\partial^2 f}{\partial y^{(1)} \partial y^{(2)}} + f \frac{\partial^2 f}{\partial y^{(1)} \partial y^{(3)}},$$

$$C = \frac{\partial^2 f}{\partial x \partial y^{(2)}} + f \frac{\partial^2 f}{\partial y^{(2)} \partial y^{(3)}},$$

$$D = \frac{\partial^2 f}{\partial x \partial y^{(3)}},$$

$$E = y^{(2)} \frac{\partial f}{\partial y} + y^{(3)} \frac{\partial f}{\partial y^{(1)}} + f \frac{\partial f}{\partial y^{(2)}} + f' \frac{\partial f}{\partial y^{(3)}},$$

$$F = \frac{\partial^2 f}{\partial x^2} + (y^{(1)})^2 \frac{\partial^2 f}{\partial y^2} + (y^{(2)})^2 \frac{\partial^2 f}{(\partial y^{(1)})^2} + (y^{(3)})^2 \frac{\partial^2 f}{(\partial y^{(2)})} + (f)^2 \frac{\partial^2 f}{(\partial y^{(3)})^2}.$$

The initial values $y_0, y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, y_0^{(iv)}$ are obtained from the given problems.

4. Numerical Examples and Results

Two non-linear fifth order problems are solved with the new method (10) to test and confirm its accuracy.

x	y_{exact}	y_{computed},	Errors in new scheme $k=6,\ h=\frac{1}{100}$
0.1000	1.115170918075647700	1.115170905970022600	1.210563e−008
0.2000	1.261402758160169700	1.261402738889510600	1.927066e−008
0.3000	1.439858807576001900	1.439858777316276100	3.025973e−008
0.4000	1.651824697641268300	1.651824650668156600	4.697311e−008
0.5000	1.898721270700125100	1.898721198617298300	7.208283e−008
0.6000	2.182118800390508500	2.182118691110499100	1.092800e−007
0.7000	2.503752707470481300	2.503752543886335900	1.635841e−007
0.8000	2.865540928492477800	2.865540686759706600	2.417328e−007
0.9000	3.269603111156967200	3.269602758482976800	3.526740e−007
1.0000	3.718281828459071300	3.718281320263648200	5.081954e−007

Table 1: Numerical solution and errors for Problem 1

Problem 1.

$$y^v = 2y'y'' - yy^{iv} - y'y''' - 8x + (x^2 - 2x - 3)e^x, \quad 0 \le x \le 1,$$

$$y(0) = 1,\ y'(0) = 1,\ y''(0) = 3,\ y'''(0) = 1,\ y^{iv}(0) = 1,\ h = 0.01.$$

Theoretical solution is y(x) = $e^x + x^2$.

Problem 2.

$$y^v = 6\{2(y')^3 + 6yy'y'' + y^2y'''\}, \quad 1 \le x \le 2,$$

$$y(1) = 1,\ y'(1) = -1,\ y''(1) = 2,\ y'''(1) = -6,\ y^{1v}(1) = 24,\ h = 0.1,\ k = 6.$$

Theoretical solution is $y(x) = \frac{1}{x}$.

5. Conclusion

This work has produced an order seven explicit continuous-coefficient method and its associated derivatives, expected to serve as a suitable predictor for an implicit method. The method is consistent, zero-stable and normalized suitable for direct integration of both non-stiff and mildly-stiff fifth order ordinary differential equations (odes) without reduction to system of lower order problems. Future research work will be directed at developing an order seven implicit method for direct implementation of fifth order odes that will make use of this method as a main predictor for better accuracy.

x	y_{exact}	y_{computed},	Errors in new scheme $k=6$, $h=\frac{1}{10}$
1.1000	0.909090909090909060	0.909090915471895360	$6.380986e{-}009$
1.2000	0.833333333333333260	0.833333503564707030	$1.702314e{-}007$
1.3000	0.769230769230769050	0.769231965619759150	$1.196389e{-}006$
1.4000	0.714285714285714080	0.714290384674668650	$4.670389e{-}006$
1.5000	0.666666666666666520	0.666679923540116400	$1.325687e{-}005$
1.6000	0.624999999999999780	0.625030829433081970	$3.082943e{-}005$
1.7000	0.588235294117646860	0.588297860465621940	$6.256635e{-}005$
1.8000	0.555555555555555360	0.555670582463722360	$1.150269e{-}004$
1.9000	0.526315789473683960	0.526512004953412100	$1.962155e{-}004$
2.0000	0.499999999999999780	0.500315637138137870	$3.156371e{-}004$

Table 2: Numerical solution and errors for Problem 2

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