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ON THE TIME-DISCRETIZATION METHOD FOR THE UNSTEADY HEAT CONDUCTION EQUATION WITH UNIFORM HEAT FLUX BOUNDARY CONDITION

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Abstract: The time-discretization method is a powerful method to provide approximate, semi-analytic solutions of parabolic differential equations in one or more dimensions. In the case of one-dimensional parabolic partial differential equations, the time-discretization method employs the two-point backward approximation for the time-derivative, while leaving the space derivative continuous. This is a simple operation that engenders a sequence of adjoint second-order ordinary differential equations, wherein the space coordinate is the independent variable and a fixed time appears as an embedded parameter. In this work, the time-discretization method is applied to the unsteady 1-D heat equation in a large plate with constant initial temperature and uniform surface heat flux as the boundary condition. Conceptually, the associated sequence of adjoint second-order ordinary differential equations of heat conduction are of quasi-stationary nature. Using the first adjoint quasi-stationary heat equation with one time jump, it is demonstrated that an approximate, semi-analytic temperature solution of good quality is easily obtainable and is valid at all time.

Key Words: time-discretization method, unsteady 1-D heat equation, uniform surface heat flux, adjoint quasi-stationary heat equations

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Nomenclature

c_v specific heat capacity at constant volume
k thermal conductivity
L semi-thickness of large plate
m^2 dimensionless time parameter, $\frac{1}{\Delta\tau}$
q_s uniform surface heat flux
t time
T temperature
T_c mid-plane temperature
T_{eq} equivalent temperature, $\frac{q_s L}{k}$
T_{in} initial temperature
T_m mean temperature
T_s surface temperature
x space coordinate
X dimensionless space coordinate, $\frac{x}{L}$

Greek letters
α thermal diffusivity, $\rho\frac{k}{c_v}$
τ dimensionless time or Fourier number, $\frac{t}{L^2/\alpha}$
Φ dimensionless temperature, $\frac{T-T_{in}}{q_s L/k}$
ρ density

1. Introduction

Unsteady heat conduction in solid bodies with various heating/cooling conditions at their surfaces constitutes problems of remarkable importance in engineering, physics and applied mathematics. When dealing with a large plate framed in Cartesian coordinates, the exact analytical solution of the unsteady 1-D heat equation with prescribed surface temperature has led to infinite series, which are double-valued functions of the space coordinate and time (Carslaw and Jaeger [1], Arpaci [2], Luikov [3], Grigull and Sanders [4], Ozis¸ik [5], Poulikakos [6] and Myers [7]). On the positive side, the infinite series possesses intrinsic good convergence, which for large dimensionless time, conveniently recede to 'one-term' series. On the negative side, the infinite series shows severe divergence at small dimensionless times, necessitating many terms to secure adequate accuracy. In fact, the prevalent behavior the infinite series becomes so abnormal for very short times that the evaluated temperatures consistently

overpredict the initial condition in contraposition with the physics of the problem. To shed light on this issue, Grigull et al. [8] evaluated the spatio-temporal temperatures in a large plate utilizing both the complete infinite series and the simplified 'one-term' series in order to compare them on a one-on-one basis. These authors concluded that the simplified 'one-term' series is accurate provided that the dimensionless time $\tau > \tau_{cr} = 0.24$ (a critical dimensionless time). The tacit discovery was achieved while permitting a relative error of 1% between the computed dimensionless temperatures at the mid-plane and the surface in the large plate. The pertinent information for τ_{cr} was first made available in the heat transfer textbook by Grigull and Sanders [4], where the mean τ_{cr} number has been reduced to $\tau_{cr}^{-} = 0.2$ for brevity. Appropriately, the validity of the 'one-term' series for practical calculations entails to a dimensionless time or Fourier number $\tau \geq 0.2$, i.e., the so-called 'large-time' solution [1-7].

Contrariwise, the exact analytical solution of the unsteady 1-D heat equation in a large plate with prescribed surface heat flux instead of prescribed surface temperature is more complicated because of two relevant factors [1-7]: 1) the dominant boundary condition at the surface is non-homogeneous and 2) no steady-state asymptotic solution exists.

One of the basic principles of applied mathematics is to split a complex problem into a reduced number of less complex sub-problems, which have been already solved or are easier to treat. In this context, the goal of the present study lies within the confines of an initial boundary value problem representative of unsteady 1-D heat equation in a large plate receiving uniform heat flux at the two surfaces.

Utilizing the two-point backward approximation for the time derivative in the unsteady 1-D heat equation, the idea behind the time-discretization method is to transform the parabolic partial differential equation into a sequence of quasi-stationary ordinary differential equations of second order accounting for an embedded time parameter. The influential time parameter is specified at a pre-set time where the temperatures in the large plate are needed.

The technical paper is divided in five parts. The first part describes the physical system and the mathematical formulation for the continuous heating of a large plate with uniform surface heat flux. A brief description of the ideas underlying the time-discretization method and its step-by-step implementation are delineated in the second part. In the third part, the time-discretization method is applied to the unsteady 1-D heat equation coupled with the imposed boundary conditions. The "all-time", exact analytic temperature distribution as taken from Luikov [3] is disclosed in the fourth part. In the fifth part, the approximate

quasi-analytic temperature results produced by the time-discretization method are compared against the exact analytic temperature results.

2. Physical System and Mathematical Formulation

The physical system deals with unsteady, unidirectional heat conduction in a large plate of thickness $2L$. The large plate has uniform initial temperature T_{in}throughout. For times $t >0$, a uniform heat flux q_s is applied at the two exposed surfaces $x = L$ and $x = -L$ in the large plate. The thermal conductivity k and the thermal diffusivity α of the material are taken as independent of temperature. Accordingly, the mathematical formulation comprises the unsteady 1-D heat equation in Cartesian coordinates

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{in} \quad 0 \leq x \leq L,\ t > 0, \tag{1}$$

subject to the initial condition

$$T\,(x, 0) = T_{in} \tag{2}$$

and the boundary conditions

$$\frac{\partial T(0,t)}{\partial x} = 0 \tag{3a}$$

$$k\frac{\partial T(L,t)}{\partial x} = q_s, \tag{3b}$$

after exploiting the symmetry of the physical system. The surface heat flux q_s showing up in eq. (3b) is commonly produced by electrical heating or radiative heating in engineering practice [6].

Within the framework of boundary conditions per se, eqs. (3a) and (3b) are classified as boundary conditions of second kind or Neumann boundary conditions after Carl Neumann [5]; more specifically eq. (3a) is homogeneous and eq. (3b) is non-homogeneous. Quantitatively, eq. (3a) implies zero heat flux (thermal symmetry) on the inner boundary $x = 0$, whereas eq. (3b) stipulates a positive heat flux on the outer right boundary $x = L$. With regards to the temperature behavior in the domain$\leq\leq$ $0\ x\ L$, it is well known that the temperatures increase linearly with time for 'long time' [1-7].

Let us introduce now dimensionless variables for the coordinate x, the time$t$ and the temperature difference $T - T_{in}$,

$$X = \frac{x}{L}, \quad \tau = \frac{t}{L^2/\alpha}, \quad \Phi = \frac{T - T_{in}}{q_s L/k}, \tag{4}$$

with respective scales L, L^2/α and q_sL/k. Notice that X is normalized, while τ and Φ are not normalized. Therefore, the unsteady 1-D heat equation (1) is transformed into

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial X^2}, \quad \text{in } 0X \leq 1, \tau > 0 \tag{5}$$

In the same way, the initial condition becomes

$$\Phi\left(X, 0\right) = 0 \tag{6},$$

and the boundary conditions turn into

$$\tau \frac{\partial \Phi\left(0,\right)}{\partial X} = 0 \tag{7a}$$

and

$$\tau \frac{\partial \Phi\left(1,\right)}{\partial X} = 1. \tag{7b}$$

3. The Time-Discretization Method

Developed by Rothe [9] more than 80 years ago, the time-discretization method embodies a powerful mathematical procedure for solving parabolic partial differential equations in one or more dimensions in approximate, semi-analytic form. Oftentimes, the time-discretization method has been called Transversal Method of Lines (Rektorys [10]).

In reference to the dimensionless unsteady 1-D heat equation (5), the time-discretization method discretizes the first-order time derivative $\dfrac{\partial \Phi}{\partial \tau}$ with the two-point backward finite-difference approximation

$$\left.\frac{\partial \Phi_p}{\partial}\right|_P \approx \frac{\Phi_p - \Phi_{p-1}}{\Delta \tau} + O\left(\Delta\right), \tag{8}$$

where $O(\Delta\tau)$ indicates truncation error of first order. Meanwhile, the second-order space derivative $\dfrac{\partial^2 \Phi}{\partial X^2}$ in eq. (5) remains continuous. To accommodate the time-discretization method, a special computational domain needs to be constructed with a set of transversal straight lines parallel to the X-coordinate and perpendicular to the τ-coordinate as sketched in Fig. 1. For easiness, the consecutive transversal straight lines are separated by equal dimensionless time

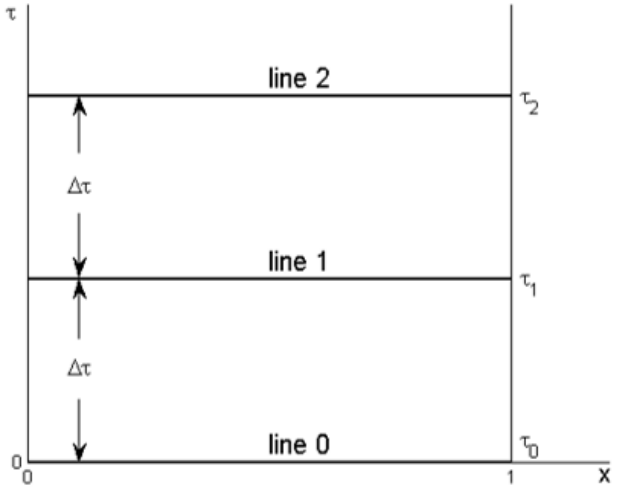

Figure 1: Computational domain for the time-discretization method

intervals $\Delta\tau$. Based on this and substituting eq. (8) into eq. (5), supplies the following sequence of adjoint second-order ordinary differential equations

$$\frac{d^2\Phi_p}{dX^2} = \frac{\Phi_p}{\Delta\tau} - \frac{\Phi_{p-1}}{\Delta\tau} \quad \text{in } 0 \leq X \leq 1. \tag{9}$$

at successive time levels τ_p ($p = 1, 2, \ldots, P$).

In a mathematical framework, the original parabolic partial differential equation (5) in the two independent variables X and τ is replaced by a sequence of adjoint second-order ordinary differential equations in the independent variable X, wherein τ is absorbed as an embedded parameter by way of a dimensionless time interval $\Delta\tau$. From physical grounds, each equation (9) for the dependent variable the dimensionless temperature Φ_p, may be viewed as a quasi-stationary heat equation associated with the dimensionless time level τ_p at the transversal line p. Correspondingly, the boundary conditions from eq. (7a) and (7b) are reformed as

$$\frac{d\Phi_p(0)}{dX} = 0 \tag{10a}$$

and

$$\frac{d\Phi_p(1)}{dX} = 1. \tag{10b}$$

The traditional finite-difference techniques for solving parabolic partial differential equations are the explicit, implicit and Crank-Nicolson methods as reported by Quarteroni and Valli [11]. As cited in [9], treating parabolic partial differential equations with the time-discretization method over those finite-difference techniques has two main advantages. One advantage is that a closed-form, semi-analytic solution is obtained instead of a numerical solution in tabulated or graphical form. The other advantage is that the truncation errors, being susceptible to the dimensionless time interval $\Delta\tau$, are easily controllable.

To begin the computational procedure, we set $p = 1$ (the first time jump) in eq. (9) linked to a small dimensionless time interval $\Delta\tau$ between the dimensionless initial time $\tau = 0$ and a future dimensionless time $\tau = \tau_1$ where the solution is needed. Therefore, the first adjoint ordinary differential equation for the dimensionless time τ_1 at the transversal line 1 is

$$\frac{d^2\Phi_1}{dX^2} - \frac{\Phi_1}{\Delta\tau} = -\frac{\Phi_0}{\Delta\tau}. \tag{11}$$

Substituting the initial condition $\Phi\,(X, 0) = \Phi_0 = 0$ from eq. (6) in eq. (11) gives rise to the homogeneous, ordinary differential heat equation of second order

$$\frac{d^2\Phi_1}{dX^2} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{12}$$

along with the respective boundary conditions

$$\frac{d\Phi_1(0)}{dX} = 0 \tag{13a}$$

and

$$\frac{d\Phi_1(1)}{dX} = 1. \tag{13b}$$

Conceptually, eq. (12) subject to eqs. (13a) and (13b) define a boundary value problem in which the subscript 1 in the dependent variable Φ_1 designates the first time jump linked to the transversal line 1 in Fig. 1.

At this point, it is convenient to define a dimensionless time parameter

$$m^2 = \frac{1}{\Delta\tau}, \tag{14}$$

so that eq. (12) is rewritten compactly as

$$\frac{d^2\Phi_1}{dX^2} - m^2\Phi_1 = 0. \tag{15}$$

4. Target Temperatures Related to Surface Heat Flux

Notwithstanding, the two most important target temperatures in the continuous heating of the large plate with uniform surface heat flux are: 1) the highest surface temperatures at any given time, i.e., $\tau\tau\Phi_{high}[ERR : md : MbegChr = 0x0028, MendChr = 0x0029, nParams = 0] = \Phi\ (1,)$ and 2) the lowest temperatures at the mid-plane at any given time, i.e., $\tau\tau\Phi_{low}[ERR : md : MbegChr = 0x0028, MendChr = 0x0029, nParams = 0] = \Phi\ (0,)$, both in dimensionless form.

Solving eq. (15), yields the approximate, semi-analytic dimensionless temperature distribution

$$\Phi_1\,(X,\,m) = \frac{1}{m \sinh\, m}\,(\cosh mX) \tag{17a}$$

or equivalently in terms of the dimensionless time τ

$$\Phi_1\,(X,\tau) = \left[\frac{\sqrt{\tau}}{\sinh\,\left(\frac{1}{\sqrt{\tau}}\right)}\right]\,\cosh\,\left(\frac{X}{\sqrt{\tau}}\right). \tag{17b}$$

From here, the approximate, semi-analytic dimensionless surface temperature passes into

$$\Phi_1\,(1,\,\tau) = \frac{\sqrt{\tau}}{\tanh\,\left(\frac{1}{\sqrt{\tau}}\right)}, \tag{18}$$

and the approximate, semi-analytic dimensionless mid-plane temperature come to be

$$\Phi_1(0,\tau) = \frac{\sqrt{\tau}}{\sinh\,\left(\frac{1}{\sqrt{\tau}}\right)}. \tag{19}$$

Furthermore, the dimensionless mean temperature is defined as

$$\Phi_{1,m}\,(\tau) = \int_0^1 \Phi_1\,(X,\tau)\ dX, \tag{20}$$

where $\Phi_1\,(X,\tau)$ is taken from eq. (17b). Performing the integration, the end product is the linear algebraic equation

$$\Phi_{1,m}\,(\tau) = \tau. \tag{20a}$$

Conversely, the solution of the so-called 0-D lumped heat equation supplies the linear algebraic equation

$$\Phi_{1,m}(\tau) = \tau, \tag{20b}$$

which is valid at all time $0 < \tau < \infty$.

5. Exact Analytic Solution for "All-Time"

The exact, analytic dimensionless temperature distribution for a large plate at "all time" as given in Luikov [3] is represented by the infinite series:

$$\Phi(X,\tau) = \tau + \left(\frac{X^2}{2} - \frac{1}{6}\right) - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\mu_n^2} \cos(\mu_n X) \exp\left(-\mu_n^2 \tau\right) \tag{21}$$

where the eigenvalues are

$$\pi\mu_n = n, \quad n = 1, 2, 3, \ldots \tag{21}$$

In general, the peculiar characteristic of the infinite series in eq. (21) is that it converges rapidly for large dimensionless time τ and in contrast diverges manifestly for small dimensionless time τ, even sometimes violating the initial condition in eq. (6). From eq. (21), the exact, analytic dimensionless temperature at the surface is

$$\Phi(1,\tau) = \tau + \frac{1}{3} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\mu_n^2} \cos(\mu_n) \exp\left(-\mu_n^2 \tau\right) \tag{22}$$

and likewise the exact, analytic dimensionless temperature at the mid-plane is

$$\Phi(0,\tau) = \tau - \frac{1}{6} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\mu_n^2} \exp\left(-\mu_n^2 \tau\right). \tag{23}$$

In addition, combining eqs. (20) and (21) gives the dimensionless mean temperature

$$\Phi_m(\tau) = \tau. \tag{24}$$

6. Approximate Analytic Solution for "Large Time"

The sequence of eigenvalues in eq. (21a) form a monotonically increasing infinite series

$$\mu_1 < \mu_2 < \mu_3 < \ldots < \mu_\infty \to \infty$$

From Sturm–Liouville theory (Hazewinkel [12]), the contribution of the first eigenvalue μ_1 toward the temperature solutions in eq. (21) is dominant after a critical dimensionless time τ_{cr} is surpassed. In other words, the direct implication is that the successive eigenvalues $\mu_2\,, \mu_3, ..., \mu_\infty$ in eq. (21a) can be discarded because their contributions are literally imperceptible. As a direct consequence, the structure of the infinite series in eq. (21) can be alleviated significantly giving way to the so-called truncated "one-term" series,

$$\Phi^{(1)}\left(X,\tau\right) = \tau + \frac{X^2}{2} - \frac{1}{6} - \frac{2}{\pi^2}\cos\left(\pi \mathrm{X}\right)\exp\left(-\pi^2\,\tau\right), \tag{25}$$

where the superscript (1) in $\Phi^{(1)}$ indicates "one term" series. To ascertain the important sub-region for "large time" characterized by $\tau \geq \tau_{cr}$ in a qualitative manner, the limit for very large time $\tau \to \infty$ is taken in the exponential term in eq. (25). Therefore, the truncated "one-term" series for "large time" abridges to the plain algebraic equation

$$\Phi^{(1)}\left(X,\tau\right) = \tau + \left(\frac{X^2}{2} - \frac{1}{6}\right), \tag{26}$$

which is decoupled in two parts: a) a linear variation with the dimensionless time τ and b) a transversal variation across the large plate described by a parabolic profile $\dfrac{X^2}{2} - \dfrac{1}{6}$. Therefore, the approximate, dimensionless analytic temperature at the surface reduces to

$$\Phi^{(1)}\left(1,\tau\right) = \tau + \frac{1}{3} \tag{27}$$

and in like manner the approximate, dimensionless analytic temperature at the mid-plane reduces to

$$\Phi^{(1)}\left(0,\tau\right) = \tau - \frac{1}{6}. \tag{28}$$

In addition, substituting eq. (25) into eq. (20) gives the dimensionless mean temperature

$$\Phi_m^{(1)}\left(\tau\right) = \tau. \tag{29}$$

In a related matter, the combination of eqs. (26) and (27) quantifies the dimensionless temperature difference that exists between the surface and the mid-plane. That is,

$$\Delta\Phi^{(1)}(\tau) = \frac{1}{2}. \tag{30}$$

7. Presentation of Approximate Results

The exact, analytic equation (21) taken from Luikov [3] constitutes the baseline solution for the spatio-temporal temperatures $\Phi(X, \tau)$ in the large plate. The associated pair of eqs. (22) and (23) was numerically evaluated at the relevant two locations, $X = 1$ and $X = 0$ with the symbolic algebra software Maple [13]. The smallest dimensionless time was set at $\tau\tau = 0.01$ with increments of $\Delta = 0.01$. As far as the number of terms is concerned,1300 terms are retained in the two infinite series of eqs. (22) and (23), allowing a relative error limit of 1%.

To characterize the sub-region dimensionless "large time" for the surface temperature in the large plate, the infinite series in eq. (22) and the linear algebraic equation (26) for "large time" are evaluated concurrently for $X = 1$ at several dimensionless time starting with a smallest $\tau = 0.01$. Employing eq. (26), the approximate dimensionless surface temperature Φ_S at $\tau = 0.2$ produces 0.5333, while the exact, analytic dimensionless surface temperatures Φ_S coming from eq. (22) provides 0.5052. Thereby, the associated relative error amounts to 5.3%. Hence, permitting a relative error of about 5%, the critical dimensionless time, $\tau_{cr} = 0.2$ could be established as a reasonable borderline for the "large time" sub-region.

7.1. Comparison Between Approximate, Semi-Analytic and Exact, Analytic Temperatures

The potent time-discretization method has been employed in this paper to obtain compact, approximate semi-analytic temperature solutions for unsteady, unidirectional heat conduction in a large plate exposed to equal uniform surface heat flux. Accordingly, the qualitative behavior of the time-discretization method will be assessed in this sub-section.

Knowingly that the two-point backward finite-difference formulation has a first order truncation error of $\tau O(\Delta)$, i.e., valid for 'short time', it is conjectured that the deduced temperature solutions $\Phi_1(X, \tau)$ should have errors

proportional to $\tau\Delta$. The computed error distribution for the dimensionless surface temperatures Φ_S in the large plate is listed in Table 1 between the dimensionless time $\tau = 0.05$ and 0.2. For instance, at $\tau = 0.05$ the error is a mere -0.029. As τ increases, the errors enlarge progressively as expected; in numbers $\tau$the error reaches -0.048 at $= 0.2$. On the other hand, the relative errors decrease gradually with the dimensionless time τ between $\tau = 0.05$ and 0.2.

Numerical evaluations of the approximate semi-analytic temperature distribution in dimensionless form for the surface in the large plate as evaluated by eq. (18) are plotted in Fig. 2. Plotted also in this figure is the approximate, semi-analytic temperature distribution in dimensionless form for the mid-plane in the large plate taken from eqs. (19). There is a commonality in the behavior of two of the three relevant dimensionless temperatures for the large plate as seen in Fig. 2. One commonality is that the dimensionless surface temperatures Φ_S obtained by the time-discretization method are slightly under-predicted with respect to the exact, analytic values.

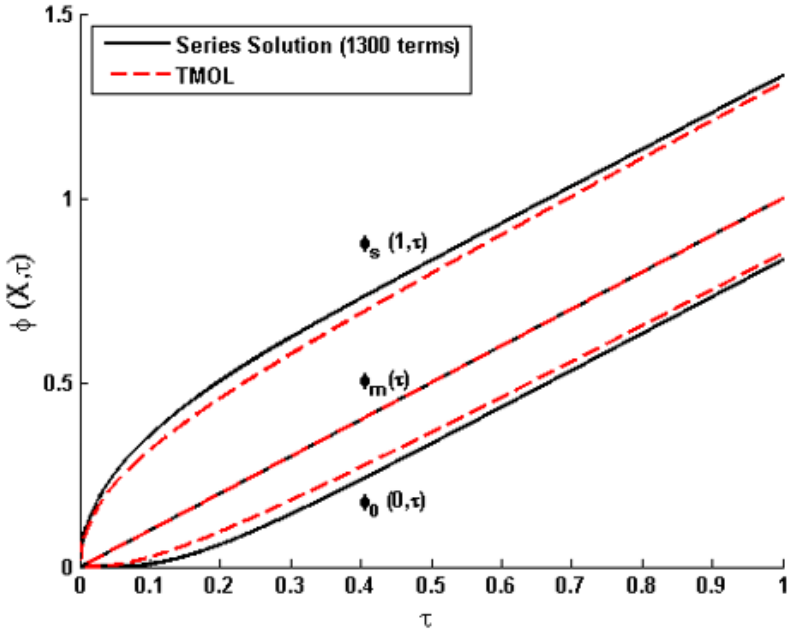

Figure 2: Dimensionless surface, mid-plane and mean temperatures: Comparison between the approximate, semi-analytic solution and the exact, analytic solution at "all time" (Luikov [3]).

The other commonality is that the dimensionless temperatures at the mid-plane Φ_0 obtained are slightly over-predicted.

When compared against the exact, analytic temperatures reported in Luikov [3], good agreement over a relatively large dimensionless time domain [0, 1] is observable in Fig. 2. In the figure, the agreement between the dimensionless temperatures supplied by two radically different solution procedures is evident for the first monotonic increasing curved line, the so-called 'short time' part and also for the second positive-sloped straight line, the so-called 'large time' part.

8. Improvement of the Approximate Temperature Results

To improve the obtained approximate temperature results, the second dimensionless time τ_2 in the transversal line 2 must be added. That is, the ensuing quasi-stationary heat equation is

$$\frac{d^2\Phi_2}{dX^2} - \frac{\Phi_2}{\Delta\tau} = -\frac{\Phi_1}{\Delta\tau}. \tag{31}$$

Herein, the appropriate initial condition $\Phi_1 = (X, \Delta\tau)$ coming from eq. (17b), provides the non-homogeneous term in eq. (31), i.e., the forcing function in other words. Hence, the ordinary differential equation of second order and non-homogeneous is

$$\frac{d^2\Phi_2}{dX^2} - \frac{\Phi_2}{\Delta\tau} = -\frac{1}{\Delta\tau}\left[\frac{\sqrt{\Delta\tau}}{\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right]\cosh\left(\frac{X}{\sqrt{\Delta\tau}}\right). \tag{32}$$

In like manner, the boundary conditions are properly rewritten as

$$\frac{d\Phi_2(0)}{dX} = 0 \tag{33a}$$

and

$$\frac{d\Phi_2(1)}{dX} = 1. \tag{33b}$$

9. Conclusions

The anticipated expectations relative to the time-discretization method for approximately solving the unsteady 1-D heat equation in the large plate are that good quality approximate temperatures should be produced for a 'short time' sub-region. In spite of this, the calculated pair of temperatures: 1) the highest temperature at the surface and 2) the lowest temperature at the mid-plane confirms that time−discretization method can be extended without harm beyond the 'short time' sub-region, namely beyond the dimensionless time $\tau \geq 0.2$. A general conclusion that may be drawn is that when using the time−discretization method with one time jump, no distinction needs to be made between the temperature solutions for three time sub-regions involving 'short time', 'intermediate time' and 'large time'. The goodness of the time-discretization method categorically demonstrated that the approximate semi-analytic temperature solutions do not necessitate more than one time jump of any size. In general, the approximate surface dimensionless temperatures obtained with the time-discretization method for the large plate lie slightly below the exact, analytic dimensionless surface temperatures.

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Appendix: Table 1. Predictions of the dimensionless surface temperature $\Phi_s(1, \tau)$ for "short time"

Dimensionless time τ	Approximate semi-analytic dimensionless temperature, eq. (18)	Exact analytic dimensionless temperature, eq. (22) *	Error (Relative error %)
0.05	0.22367	0.25231	-0.029 (11.3)
0.10	0.31736	0.35683	-0.039 (10.9)
0.15	0.39175	0.43709	-0.045 (10.4)
0.20	0.45755	0.50517	-0.048 (9.4)

* infinite series evaluated with 1300 terms.