International Journal of Differential Equations and Applications
Volume 14 No. 2 2015, 81-99
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v14i2.2092

acadpubl.eu

THE POWER SERIES EXPANSION OF MATHIEU FUNCTION AND ITS INTEGRAL FORMALISM

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Abstract: By applying three term recurrence formula (Choun, 2012 [7]), power series expansions in closed forms of Mathieu equation for an infinite series and their integral forms are constructed.

One interesting observation resulting from the calculations is the fact that a modified Bessel function recurs in each of sub-integral forms: the first sub-integral form contains zero term of $A_n's$, the second one contains one term of A_n's, the third one contains two terms of A_n's, etc. Section 5 contains two additional examples of Mathieu equation.

This paper is 5th out of 10 in series "Special functions and three term recurrence formula (3TRF)." See section 6 for all the papers in the series.

AMS Subject Classification: 33E10, 34B30
Key Words: Mathieu equation, integral form, three term recurrence formula, modified Bessel function

1. Introduction

Mathieu ordinary differential equation is of Fuchsian types with the two regular and one irregular singularities. In contrast, Heun equation of Fuchsian types has the four regular singularities. Heun equation has the four kind of conflu-

Received: November 26, 2014

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ent forms: (slowromancapi@) confluent Heun (two regular and one irregular singularities), (slowromancapii@) doubly confluent Heun (two irregular singularities), (slowromancapiii@) biconfluent Heun (one regular and one irregular singularities) and (slowromancapiv@) triconfluent Heun equations (one irregular singularity). For DLFM version [25], Mathieu equation in algebraic forms is also derived from the confluent Heun equation by changing all coefficients $\delta = \gamma \to \frac{1}{2}$, $\epsilon \to 0$, $\alpha \to q$ and $q \to \frac{\lambda+2q}{4}$.

One example of three therm recursion relations is the Mathieu equation, introduced by Émile Léonard Mathieu (1868)[22], while investigating the vibrating elliptical drumhead. Mathieu equation, known for the elliptic cylinder equation, appears in diverse areas such as astronomy and physical problems involving Schrödinger equation for a periodic potentials [17], the parametric Resonance in the reheating process of universe [32], and wave equations in general relativity[4], etc. Mathieu function has been used in various areas in modern physics and mathematics. [1, 2, 3, 18, 20, 23, 26, 27, 29]

Unfortunately, even though Mathieu equation has been observed in various areas mentioned above, there are no power series expansions in closed forms and their integral formalism analytically. The Mathieu functions have only been described in numerical approximations (Whittaker 1914[30], Frenkel and Portugal 2001[19]). Sips 1949[28], Frenkel and Portugal 2001[19] argued that it is not possible to represent analytically the Mathieu functions in a simple and handy way.

By applying the same method I used in analyzing Heun functions [8, 9], I construct power series expansions of Mathieu equation in closed forms and their integral representations analytically using three-term recurrence formula (3TRF) [7].

Mathieu equation is written by

$$\frac{d^2y}{dz^2} + (\lambda - 2q\,\cos 2z)\,y = 0 \tag{1}$$

where λ and q are parameters. This is an equation with periodic-function coefficient. Mathieu equation also can be described in algebraic forms putting $x = \cos^2 z$:

$$4x(1-x)\frac{d^2y}{dx^2} + 2(1-2x)\frac{dy}{dx} + (\lambda + 2q - 4qx)y = 0 \tag{2}$$

This equation has two regular singularities: $x = 0$ and $x = 1$; the other singularity $x = \infty$ is irregular. Assume that its solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\nu} \tag{3}$$

where ν is an indicial root. Plug (3) into (2).

$$c_{n+1} = A_n\, c_n + B_n\, c_{n-1} \qquad ; n \geq 1 \tag{4}$$

where,

$$A_n = \frac{4(n+\nu)^2 - (\lambda + 2q)}{2(n+1+\nu)(2(n+\nu)+1)} \tag{5a}$$

$$B_n = \frac{4q}{2(n+1+\nu)(2(n+\nu)+1)} \tag{5b}$$

$$c_1 = A_0\, c_0 \tag{5c}$$

We have two indicial roots which are $\nu = 0$ and $\frac{1}{2}$.

2. Power series expansion for infinite series

By putting a power series $y(x) = \sum_{n=0}^{\infty} c_n x^{n+\nu}$ into linear ODEs, the recurrence relation between successive coefficients starts to appear. In general, the recurrence relation for a 3-term is given by (4) where $c_1 = A_0\, c_0$ and $c_0 \neq 0$. There are an infinite series and three types of polynomials in three-term recurrence relation of a linear ordinary differential equation: (slowromancapi@) polynomial which makes B_n term terminated: A_n term is not terminated, designated as 'a polynomial of type 1,' (slowromancapii@) polynomial which makes A_n term terminated: B_n term is not terminated, denominated as 'a polynomial of type 2' and (slowromancapiii@) polynomial which makes A_n and B_n terms terminated at the same time, referred as 'a complete polynomial.'

For $n = 0, 1, 2, 3, \cdots$ in (4), the sequence c_n is expanded to combinations of A_n and B_n terms. I suggest that a sub-power series $y_l(x)$ where $l = 0, 1, 2, 3, \cdots$ is constructed by observing the term of sequence c_n which includes l terms of $A_n's$ [7]. The power series solution is described by sums of each $y_l(x)$ such as $y(x) = \sum_{n=0}^{\infty} y_n(x)$. By allowing A_n in the sequence c_n is the leading term of each sub-power series $y_l(x)$, the general summation formulas of the 3-term recurrence relation in a linear ODE are constructed for an infinite series and a polynomial of type 1, designated as 'three term recurrence formula (3TRF).'

As we see (5b), there is no way to make B_n term terminated at certain value of an index summation n. Because the numerator of (5b) is just consist of a constant q parameter. So there are only two kind of power series expansions such as an infinite series and a polynomial of type 2. In this paper I construct an analytic solution of Mathieu equation for an infinite series. And its local solutions for a polynomial of type 2 are available in chapter 7 of Ref.[16].

In Ref.[7] the general expression of a power series $y(x)$ for an infinite series is defined by

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\
&= c_0 \Bigg\{ \sum_{i_0=0}^{\infty} \left(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \right) x^{2i_0+\lambda} \\
&+ \sum_{i_0=0}^{\infty} \left\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\infty} \left(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \right) \right\} x^{2i_2+1+\lambda} \\
&+ \sum_{N=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\infty} \Bigg\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \\
&\times \prod_{k=1}^{N-1} \left(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \right) \\
&\times \sum_{i_{2N}=i_{2(N-1)}}^{\infty} \left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \right) \Bigg\} \Bigg\} x^{2i_{2N}+N+\lambda} \Bigg\} \qquad (6)
\end{aligned}$$

Substitute (5a)-(5c) into (6). In this article Pochhammer symbol $(x)_n$ is used to represent the rising factorial: $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. The general expression of a power series of Mathieu equation for an infinite series is given by

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\
&= c_0 x^{\nu} \Bigg\{ \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0} (\frac{3}{4}+\frac{\nu}{2})_{i_0}} \eta^{i_0} \\
&+ \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2}+\frac{\nu}{2})(i_0+\frac{1}{4}+\frac{\nu}{2})} \frac{1}{(1+\frac{\nu}{2})_{i_0} (\frac{3}{4}+\frac{\nu}{2})_{i_0}} \\
&\times \sum_{i_1=i_0}^{\infty} \frac{(\frac{3}{2}+\frac{\nu}{2})_{i_0} (\frac{5}{4}+\frac{\nu}{2})_{i_0}}{(\frac{3}{2}+\frac{\nu}{2})_{i_1} (\frac{5}{4}+\frac{\nu}{2})_{i_1}} \eta^{i_1} \Bigg\} x \\
&+ \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2}+\frac{\nu}{2})(i_0+\frac{1}{4}+\frac{\nu}{2})} \frac{1}{(1+\frac{\nu}{2})_{i_0} (\frac{3}{4}+\frac{\nu}{2})_{i_0}}
\end{aligned}$$

$$\times \prod_{k=1}^{n-1}\left\{\sum_{i_k=i_{k-1}}^{\infty}\frac{(i_k+\frac{k}{2}+\frac{\nu}{2})^2-\frac{1}{4^2}(\lambda+2q)}{(i_k+\frac{k}{2}+\frac{1}{2}+\frac{\nu}{2})(i_k+\frac{k}{2}+\frac{1}{4}+\frac{\nu}{2})}\right.$$
$$\times \left.\frac{(1+\frac{k}{2}+\frac{\nu}{2})_{i_{k-1}}(\frac{3}{4}+\frac{k}{2}+\frac{\nu}{2})_{i_{k-1}}}{(1+\frac{k}{2}+\frac{\nu}{2})_{i_k}(\frac{3}{4}+\frac{k}{2}+\frac{\nu}{2})_{i_k}}\right\}$$
$$\times \left.\sum_{i_n=i_{n-1}}^{\infty}\frac{(1+\frac{n}{2}+\frac{\nu}{2})_{i_{n-1}}(\frac{3}{4}+\frac{n}{2}+\frac{\nu}{2})_{i_{n-1}}}{(1+\frac{n}{2}+\frac{\nu}{2})_{i_n}(\frac{3}{4}+\frac{n}{2}+\frac{\nu}{2})_{i_n}}\eta^{i_n}\right\}x^n\Bigg\} \tag{7}$$

where $\eta=\frac{1}{4}qx^2$. Put $c_0=1$ as $\nu=0$ for the first independent solution of Mathieu equation and $\nu=\frac{1}{2}$ for the second one in (7).

Remark 1. The power series expansion of Mathieu equation of the first kind for an infinite series about $x=0$ using 3TRF is given by

$$\begin{aligned} y(x) &= MF\left(q,\lambda;\eta=\frac{1}{4}qx^2,x=\cos^2 z\right)\\ &= \sum_{i_0=0}^{\infty}\frac{1}{(1)_{i_0}(\frac{3}{4})_{i_0}}\eta^{i_0}\\ &+ \left\{\sum_{i_0=0}^{\infty}\frac{i_0^2-\frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2})(i_0+\frac{1}{4})}\frac{1}{(1)_{i_0}(\frac{3}{4})_{i_0}}\sum_{i_1=i_0}^{\infty}\frac{(\frac{3}{2})_{i_0}(\frac{5}{4})_{i_0}}{(\frac{3}{2})_{i_1}(\frac{5}{4})_{i_1}}\eta^{i_1}\right\}x \end{aligned}$$

$$+ \sum_{n=2}^{\infty}\left\{\sum_{i_0=0}^{\infty}\frac{i_0^2-\frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2})(i_0+\frac{1}{4})}\frac{1}{(1)_{i_0}(\frac{3}{4})_{i_0}}\right.$$
$$\times \prod_{k=1}^{n-1}\left\{\sum_{i_k=i_{k-1}}^{\infty}\frac{(i_k+\frac{k}{2})^2-\frac{1}{4^2}(\lambda+2q)}{(i_k+\frac{k}{2}+\frac{1}{2})(i_k+\frac{k}{2}+\frac{1}{4})}\frac{(1+\frac{k}{2})_{i_{k-1}}(\frac{3}{4}+\frac{k}{2})_{i_{k-1}}}{(1+\frac{k}{2})_{i_k}(\frac{3}{4}+\frac{k}{2})_{i_k}}\right\}$$
$$\times \left.\sum_{i_n=i_{n-1}}^{\infty}\frac{(1+\frac{n}{2})_{i_{n-1}}(\frac{3}{4}+\frac{n}{2})_{i_{n-1}}}{(1+\frac{n}{2})_{i_n}(\frac{3}{4}+\frac{n}{2})_{i_n}}\eta^{i_n}\right\}x^n \tag{8}$$

Remark 2. The power series expansion of Mathieu equation of the second kind for an infinite series about $x=0$ using 3TRF is given by

$$y(x)=MS\left(q,\lambda;\eta=\frac{1}{4}qx^2,x=\cos^2 z\right)$$
$$= x^{\frac{1}{2}}\left\{\sum_{i_0=0}^{\infty}\frac{1}{(\frac{5}{4})_{i_0}(1)_{i_0}}\eta^{i_0}\right.$$

$$
\begin{aligned}
&+ \quad \left\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{1}{4})^2-\frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{3}{4})(i_0+\frac{1}{2})} \frac{1}{(\frac{5}{4})_{i_0}(1)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(\frac{7}{4})_{i_0}(\frac{3}{2})_{i_0}}{(\frac{7}{4})_{i_1}(\frac{3}{2})_{i_1}} \eta^{i_1} \right\} x \\
&+ \quad \sum_{n=2}^{\infty} \left\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{1}{4})^2-\frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{3}{4})(i_0+\frac{1}{2})} \frac{1}{(\frac{5}{4})_{i_0}(1)_{i_0}} \right. \\
&\times \quad \prod_{k=1}^{n-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+\frac{1}{4}+\frac{k}{2})^2-\frac{1}{4^2}(\lambda+2q)}{(i_k+\frac{3}{4}+\frac{k}{2})(i_k+\frac{1}{2}+\frac{k}{2})} \frac{(\frac{5}{4}+\frac{k}{2})_{i_{k-1}}(1+\frac{k}{2})_{i_{k-1}}}{(\frac{5}{4}+\frac{k}{2})_{i_k}(1+\frac{k}{2})_{i_k}} \right\} \\
&\times \quad \left. \sum_{i_n=i_{n-1}}^{\infty} \frac{(\frac{5}{4}+\frac{n}{2})_{i_{n-1}}(1+\frac{n}{2})_{i_{n-1}}}{(\frac{5}{4}+\frac{n}{2})_{i_n}(1+\frac{n}{2})_{i_n}} \eta^{i_n} \right\} x^n \Bigg\}
\end{aligned} \tag{9}
$$

3. Integral formalism for infinite series

I consider the definite integral representations of Mathieu equation by applying 3TRF. There is a generalized hypergeometric function such as

$$
\begin{aligned}
G_j &= \sum_{i_j=i_{j-1}}^{\infty} \frac{(1+\frac{j}{2}+\frac{\nu}{2})_{i_{j-1}}(\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_{i_{j-1}}}{(1+\frac{j}{2}+\frac{\nu}{2})_{i_j}(\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_{i_j}} \eta^{i_j} \\
&= \sum_{l=0}^{\infty} \frac{\eta^{l+i_{j-1}}}{(i_{j-1}+\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_l \, (i_{j-1}+1+\frac{j}{2}+\frac{\nu}{2})_l} \\
&= \frac{\eta^l}{\aleph_{i_{j-1}}} \sum_{l=0}^{\infty} \frac{B^{(1)}_{i_{j-1}} B^{(2)}_{i_{j-1}} \eta^{i_{j-1}}}{(1)_l \, l!}
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
B^{(1)}_{i_{j-1}} &= B(i_{j-1}-\frac{1}{4}+\frac{j}{2}+\frac{\nu}{2}, l+1) \\
B^{(2)}_{i_{j-1}} &= B(i_{j-1}+\frac{j}{2}+\frac{\nu}{2}, l+1) \\
\aleph_{i_{j-1}} &= \frac{1}{(i_{j-1}-\frac{1}{4}+\frac{j}{2}+\frac{\nu}{2})(i_{j-1}+\frac{j}{2}+\frac{\nu}{2})}
\end{aligned}
$$

By using integral form of beta function,

$$
B^{(1)}_{i_{j-1}} = \int_0^1 dt_j \, t_j^{i_{j-1}-\frac{5}{4}+\frac{j}{2}+\frac{\nu}{2}} (1-t_j)^l \tag{12a}
$$

$$B^{(2)}_{i_{j-1}} = \int_0^1 du_j \; u_j^{i_{j-1}-1+\frac{j}{2}+\frac{\nu}{2}}(1-u_j)^l \tag{12b}$$

Substitute (12a) and (12b) into (10). And multiply $\aleph_{i_{j-1}}$ into the new (10).

$$\begin{aligned} K_j &= \aleph_{i_{j-1}} \sum_{i_j=i_{j-1}}^{\infty} \frac{(1+\frac{j}{2}+\frac{\nu}{2})_{i_{j-1}}(\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_{i_{j-1}}}{(1+\frac{j}{2}+\frac{\nu}{2})_{i_j}(\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_{i_j}}\eta^{i_j} \\ &= \int_0^1 dt_j \; t_j^{-\frac{5}{4}+\frac{j}{2}+\frac{\nu}{2}} \int_0^1 du_j \; \frac{(\eta t_j u_j)^{i_{j-1}}}{u_j^{1-\frac{j}{2}-\frac{\nu}{2}}} \sum_{l=0}^{\infty} \frac{\big(\eta(1-t_j)(1-u_j)\big)^l}{(1)_l \; l!} \end{aligned} \tag{13}$$

The modified Bessel function is defined by

$$\begin{aligned} I_\alpha(x) &= \sum_{l=0}^{\infty} \frac{1}{l!\;(l+\alpha)!}\left(\frac{x}{2}\right)^{2l+\alpha} \\ &= \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} \int_{-1}^1 dv_j \; (1-v_j^2)^{\alpha-\frac{1}{2}} e^{-xv_j} \end{aligned} \tag{14}$$

replaced α and x by 0 and $2\sqrt{\eta(1-t_j)(1-u_j)}$ in (14).

$$\begin{aligned} I_0\left(2\sqrt{\eta(1-t_j)(1-u_j)}\right) &= \sum_{l=0}^{\infty} \frac{1}{l!\;(1)_l}\big(\eta(1-t_j)(1-u_j)\big)^l \\ &= \frac{1}{\pi}\int_{-1}^1 dv_j \; (1-v_j^2)^{-\frac{1}{2}} \exp\left(-2\sqrt{\eta(1-t_j)(1-u_j)}\; v_j\right) \end{aligned} \tag{15}$$

Substitute (15) into (13).

$$\begin{aligned} K_j &= \aleph_{i_{j-1}} \sum_{i_j=i_{j-1}}^{\infty} \frac{(1+\frac{j}{2}+\frac{\nu}{2})_{i_{j-1}}(\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_{i_{j-1}}}{(1+\frac{j}{2}+\frac{\nu}{2})_{i_j}(\frac{3}{4}+\frac{j}{2}+\frac{\nu}{2})_{i_j}}\eta^{i_j} \\ &= \int_0^1 dt_j \; t_j^{-\frac{5}{4}+\frac{j}{2}+\frac{\nu}{2}} \int_0^1 du_j \; u_j^{-1+\frac{j}{2}+\frac{\nu}{2}} \\ &\times \; I_0\left(2\sqrt{\eta(1-t_j)(1-u_j)}\right)(\eta t_j u_j)^{i_{j-1}} \end{aligned} \tag{16}$$

where

$$\aleph_{i_{j-1}} = \frac{1}{(i_{j-1}-\frac{1}{4}+\frac{j}{2}+\frac{\nu}{2})(i_{j-1}+\frac{j}{2}+\frac{\nu}{2})}$$

Substitute (16) into (7): apply K_1 into the second summation of sub-power series $y_1(x)$; apply K_2 into the third summation and K_1 into the second summation of sub-power series $y_2(x)$; apply K_3 into the forth summation, K_2 into the third summation and K_1 into the second summation of sub-power series $y_3(x)$, etc.[1]

Theorem 3. *The general expression of the integral representation of Mathieu equation for an infinite series is given by*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\ &= c_0 x^{\nu} \left\{ \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0} (\frac{3}{4}+\frac{\nu}{2})_{i_0}} \eta^{i_0} \right. \\ &+ \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\, t_{n-k}^{-\frac{5}{4}+\frac{1}{2}(n-k+\nu)} \int_0^1 du_{n-k}\, u_{n-k}^{-1+\frac{1}{2}(n-k+\nu)} \right. \right. \\ &\times I_0 \left(2\sqrt{w_{n+1-k,n}(1-t_{n-k})(1-u_{n-k})} \right) \\ &\times \left. \left(w_{n-k,n}^{-\frac{1}{2}(n-1-k+\nu)} \left(w_{n-k,n} \partial_{w_{n-k,n}} \right)^2 w_{n-k,n}^{\frac{1}{2}(n-1-k+\nu)} - \frac{1}{4^2}(\lambda+2q) \right) \right\} \\ &\times \left. \left. \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0} (\frac{3}{4}+\frac{\nu}{2})_{i_0}} w_{1,n}^{i_0} \right\} x^n \right\} \end{aligned} \tag{17}$$

where

$$w_{a,b} = \begin{cases} \eta \displaystyle\prod_{l=a}^{b} t_l u_l \;\; \textit{where } a \le b \\ \eta \;\; \textit{only if } a > b \end{cases} \tag{18}$$

Proof. In (7) sub-power series $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$ of Mathieu equation for an infinite series are given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \tag{19}$$

[1] $y_1(x)$ means the sub-power series in (7) contains one term of $A_n's$, $y_2(x)$ means the sub-power series in (7) contains two terms of $A_n's$, $y_3(x)$ means the sub-power series in (7) contains three terms of $A_n's$, etc.

where

$$y_0(x) = c_0 x^\nu \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} \eta^{i_0} \tag{20a}$$

$$\begin{aligned} y_1(x) &= c_0 x^\nu \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2}+\frac{\nu}{2})(i_0+\frac{1}{4}+\frac{\nu}{2})} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} \\ &\times \sum_{i_1=i_0}^{\infty} \frac{(\frac{3}{2}+\frac{\nu}{2})_{i_0}(\frac{5}{4}+\frac{\nu}{2})_{i_0}}{(\frac{3}{2}+\frac{\nu}{2})_{i_1}(\frac{5}{4}+\frac{\nu}{2})_{i_1}} \eta^{i_1} \Bigg\} x \end{aligned} \tag{20b}$$

$$\begin{aligned} y_2(x) &= c_0 x^\nu \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2}+\frac{\nu}{2})(i_0+\frac{1}{4}+\frac{\nu}{2})} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} \\ &\times \sum_{i_1=i_0}^{\infty} \frac{(i_1+\frac{1}{2}+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_1+1+\frac{\nu}{2})(i_1+\frac{3}{4}+\frac{\nu}{2})} \frac{(\frac{3}{2}+\frac{\nu}{2})_{i_0}(\frac{5}{4}+\frac{\nu}{2})_{i_0}}{(\frac{3}{2}+\frac{\nu}{2})_{i_1}(\frac{5}{4}+\frac{\nu}{2})_{i_1}} \\ &\times \sum_{i_2=i_1}^{\infty} \frac{(2+\frac{\nu}{2})_{i_1}(\frac{7}{4}+\frac{\nu}{2})_{i_1}}{(2+\frac{\nu}{2})_{i_2}(\frac{7}{4}+\frac{\nu}{2})_{i_2}} \eta^{i_2} \Bigg\} x^2 \end{aligned} \tag{20c}$$

$$\begin{aligned} y_3(x) &= c_0 x^\nu \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2}+\frac{\nu}{2})(i_0+\frac{1}{4}+\frac{\nu}{2})} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} \\ &\times \sum_{i_1=i_0}^{\infty} \frac{(i_1+\frac{1}{2}+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_1+1+\frac{\nu}{2})(i_1+\frac{3}{4}+\frac{\nu}{2})} \frac{(\frac{3}{2}+\frac{\nu}{2})_{i_0}(\frac{5}{4}+\frac{\nu}{2})_{i_0}}{(\frac{3}{2}+\frac{\nu}{2})_{i_1}(\frac{5}{4}+\frac{\nu}{2})_{i_1}} \\ &\times \sum_{i_2=i_1}^{\infty} \frac{(i_2+1+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_2+\frac{3}{2}+\frac{\nu}{2})(i_2+\frac{5}{4}+\frac{\nu}{2})} \frac{(2+\frac{\nu}{2})_{i_1}(\frac{7}{4}+\frac{\nu}{2})_{i_1}}{(2+\frac{\nu}{2})_{i_2}(\frac{7}{4}+\frac{\nu}{2})_{i_2}} \\ &\times \sum_{i_3=i_2}^{\infty} \frac{(\frac{5}{2}+\frac{\nu}{2})_{i_2}(\frac{9}{4}+\frac{\nu}{2})_{i_2}}{(\frac{5}{2}+\frac{\nu}{2})_{i_3}(\frac{9}{4}+\frac{\nu}{2})_{i_3}} \eta^{i_3} \Bigg\} x^3 \end{aligned} \tag{20d}$$

Put $j = 1$ in (16). Take the new (16) into (20b).

$$\begin{aligned} y_1(x) &= c_0 x^{\nu+1} \int_0^1 dt_1\, t_1^{-\frac{3}{4}+\frac{\nu}{2}} \int_0^1 du_1\, u_1^{-\frac{1}{2}+\frac{\nu}{2}} I_0\left(2\sqrt{\eta(1-t_1)(1-u_1)}\right) \\ &\times \sum_{i_0=0}^{\infty} \left(\left(i_0+\frac{\nu}{2}\right)^2 - \frac{1}{4^2}(\lambda+2q) \right) \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} (\eta t_1 u_1)^{i_0} \end{aligned}$$

$$
\begin{aligned}
&= c_0 x^{\nu+1} \int_0^1 dt_1\, t_1^{-\frac{3}{4}+\frac{\nu}{2}} \int_0^1 du_1\, u_1^{-\frac{1}{2}+\frac{\nu}{2}} I_0\left(2\sqrt{\eta(1-t_1)(1-u_1)}\right) \\
&\times \left(w_{1,1}^{-\frac{\nu}{2}} \left(w_{1,1}\partial_{w_{1,1}}\right)^2 w_{1,1}^{\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q)\right) \\
&\times \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} w_{1,1}^{i_0} \qquad (21)
\end{aligned}
$$

where $w_{1,1} = \eta \prod_{l=1}^{1} t_l u_l$. Put $j = 2$ in (16). Take the new (16) into (20c).

$$
\begin{aligned}
y_2(x) &= c_0 x^{\nu+2} \int_0^1 dt_2\, t_2^{-\frac{1}{4}+\frac{\nu}{2}} \int_0^1 du_2\, u_2^{\frac{\nu}{2}} I_0\left(2\sqrt{\eta(1-t_2)(1-u_2)}\right) \\
&\times \left(w_{2,2}^{-(\frac{1}{2}+\frac{\nu}{2})} \left(w_{2,2}\partial_{w_{2,2}}\right)^2 w_{2,2}^{\frac{1}{2}+\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q)\right) \\
&\times \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\nu}{2})^2 - \frac{1}{4^2}(\lambda+2q)}{(i_0+\frac{1}{2}+\frac{\nu}{2})(i_0+\frac{1}{4}+\frac{\nu}{2})} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} \\
&\times \sum_{i_1=i_0}^{\infty} \frac{(\frac{3}{2}+\frac{\nu}{2})_{i_0}(\frac{5}{4}+\frac{\nu}{2})_{i_0}}{(\frac{3}{2}+\frac{\nu}{2})_{i_1}(\frac{5}{4}+\frac{\nu}{2})_{i_1}} w_{2,2}^{i_1} \qquad (22)
\end{aligned}
$$

where $w_{2,2} = \eta \prod_{l=2}^{2} t_l u_l$. Put $j = 1$ and $\eta = w_{2,2}$ in (16). Take the new (16) into (22).

$$
\begin{aligned}
y_2(x) &= c_0 x^{\nu+2} \int_0^1 dt_2\, t_2^{-\frac{1}{4}+\frac{\nu}{2}} \int_0^1 du_2\, u_2^{\frac{\nu}{2}} I_0\left(2\sqrt{\eta(1-t_2)(1-u_2)}\right) \\
&\times \left(w_{2,2}^{-(\frac{1}{2}+\frac{\nu}{2})} \left(w_{2,2}\partial_{w_{2,2}}\right)^2 w_{2,2}^{\frac{1}{2}+\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q)\right) \\
&\times \int_0^1 dt_1\, t_1^{-\frac{3}{4}+\frac{\nu}{2}} \int_0^1 du_1\, u_1^{-\frac{1}{2}+\frac{\nu}{2}} I_0\left(2\sqrt{w_{2,2}(1-t_1)(1-u_1)}\right) \\
&\times \left(w_{1,2}^{-\frac{\nu}{2}} \left(w_{1,2}\partial_{w_{1,2}}\right)^2 w_{1,2}^{\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q)\right) \\
&\times \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} w_{1,2}^{i_0} \qquad (23)
\end{aligned}
$$

where $w_{1,2} = \eta \prod_{l=1}^{2} t_l u_l$. By using similar process for the previous cases of integral forms of $y_1(x)$ and $y_2(x)$, the integral form of sub-power series expansion of $y_3(x)$ is

$$
y_3(x) = c_0 x^{\nu+3} \int_0^1 dt_3\, t_3^{\frac{1}{4}+\frac{\nu}{2}} \int_0^1 du_3\, u_3^{\frac{1}{2}+\frac{\nu}{2}} I_0\left(2\sqrt{\eta(1-t_3)(1-u_3)}\right)
$$

$$
\begin{aligned}
&\times \left(w_{3,3}^{-(1+\frac{\nu}{2})} \left(w_{3,3}\partial_{w_{3,3}} \right)^2 w_{3,3}^{1+\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q) \right) \\
&\times \int_0^1 dt_2\, t_2^{-\frac{1}{4}+\frac{\nu}{2}} \int_0^1 du_2\, u_2^{\frac{\nu}{2}} I_0\left(2\sqrt{w_{3,3}(1-t_2)(1-u_2)} \right) \\
&\times \left(w_{2,3}^{-(\frac{1}{2}+\frac{\nu}{2})} \left(w_{2,3}\partial_{w_{2,3}} \right)^2 w_{2,3}^{\frac{1}{2}+\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q) \right) \\
&\times \int_0^1 dt_1\, t_1^{-\frac{3}{4}+\frac{\nu}{2}} \int_0^1 du_1\, u_1^{-\frac{1}{2}+\frac{\nu}{2}} I_0\left(2\sqrt{w_{2,3}(1-t_1)(1-u_1)} \right) \\
&\times \left(w_{1,3}^{-\frac{\nu}{2}} \left(w_{1,3}\partial_{w_{1,3}} \right)^2 w_{1,3}^{\frac{\nu}{2}} - \frac{1}{4^2}(\lambda+2q) \right) \\
&\times \sum_{i_0=0}^{\infty} \frac{1}{(1+\frac{\nu}{2})_{i_0}(\frac{3}{4}+\frac{\nu}{2})_{i_0}} w_{1,3}^{i_0}
\end{aligned} \tag{24}
$$

where

$$
w_{3,3} = \eta \prod_{l=3}^{3} t_l u_l \qquad w_{2,3} = \eta \prod_{l=2}^{3} t_l u_l \qquad w_{1,3} = \eta \prod_{l=1}^{3} t_l u_l
$$

By repeating this process for all higher terms of integral forms of sub-summation $y_m(x)$ terms where $m \geq 4$, we obtain every integral forms of $y_m(x)$ terms. Since we substitute (20a), (21), (23), (24) and including all integral forms of $y_m(x)$ terms where $m \geq 4$ into (19), we obtain (17). □

Put $c_0 = 1$ and $\nu = 0$ in (17).

$$
\begin{aligned}
y(x) &= MF\left(q, \lambda; x = \cos^2 z, \eta = \frac{1}{4}qx^2 \right) \\
&= \sum_{i_0=0}^{\infty} \frac{1}{(1)_{i_0}(\frac{3}{4})_{i_0}} \eta^{i_0} \\
&+ \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\, t_{n-k}^{-\frac{5}{4}+\frac{1}{2}(n-k)} \int_0^1 du_{n-k}\, u_{n-k}^{-1+\frac{1}{2}(n-k)} \right.\right. \\
&\times I_0\left(2\sqrt{w_{n+1-k,n}(1-t_{n-k})(1-u_{n-k})} \right) \\
&\times \left.\left(w_{n-k,n}^{-\frac{1}{2}(n-1-k)} \left(w_{n-k,n}\partial_{w_{n-k,n}} \right)^2 w_{n-k,n}^{\frac{1}{2}(n-1-k)} - \frac{1}{4^2}(\lambda+2q) \right) \right\} \\
&\times \left. \sum_{i_0=0}^{\infty} \frac{1}{(1)_{i_0}(\frac{3}{4})_{i_0}} w_{1,n}^{i_0} \right\} x^n
\end{aligned} \tag{25}
$$

replaced α and x by $-\frac{1}{4}$ and $2\sqrt{\eta}$ in (14).

$$I_{-\frac{1}{4}}\left(2\sqrt{\eta}\right)=\frac{\eta^{-\frac{1}{8}}}{\Gamma(\frac{3}{4})}\sum_{l=0}^{\infty}\frac{1}{l!\ (\frac{3}{4})_l}\eta^l$$

$$=\frac{\eta^{-\frac{1}{8}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\int_{-1}^{1}dv_0\ (1-v_0^2)^{-\frac{3}{4}}\exp\left(-2\sqrt{\eta}\ v_0\right) \tag{26}$$

Similarly,

$$I_{-\frac{1}{4}}\left(2\sqrt{w_{1,n}}\right)=\frac{w_{1,n}^{-\frac{1}{8}}}{\Gamma(\frac{3}{4})}\sum_{l=0}^{\infty}\frac{1}{l!\ (\frac{3}{4})_l}w_{1,n}^{l}$$

$$=\frac{w_{1,n}^{-\frac{1}{8}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\int_{-1}^{1}dv_0\ (1-v_0^2)^{-\frac{3}{4}}\exp\left(-2\sqrt{w_{1,n}}\ v_0\right) \tag{27}$$

Substitute (26) and (27) into (25).

Remark 4. The integral representation of Mathieu equation of the first kind for an infinite series about $x=0$ using 3TRF is given by

$$\begin{aligned}
y(x) &= MF\left(q,\lambda;x=\cos^2 z,\eta=\frac{1}{4}qx^2\right)\\
&= \Gamma\left(3/4\right)\left\{\eta^{\frac{1}{8}}I_{-\frac{1}{4}}\left(2\sqrt{\eta}\right)\right.\\
&+ \sum_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\ t_{n-k}^{-\frac{5}{4}+\frac{1}{2}(n-k)}\int_0^1 du_{n-k}\ u_{n-k}^{-1+\frac{1}{2}(n-k)}\right.\right.\\
&\times I_0\left(2\sqrt{w_{n+1-k,n}(1-t_{n-k})(1-u_{n-k})}\right)\\
&\times \left.\left(w_{n-k,n}^{-\frac{1}{2}(n-1-k)}\left(w_{n-k,n}\partial_{w_{n-k,n}}\right)^2 w_{n-k,n}^{\frac{1}{2}(n-1-k)}-\frac{1}{4^2}(\lambda+2q)\right)\right\}\\
&\times \left.w_{1,n}^{\frac{1}{8}}I_{-\frac{1}{4}}\left(2\sqrt{w_{1,n}}\right)\right\}x^n\Bigg\}
\end{aligned} \tag{28}$$

Put $c_0=1$ and $\nu=\frac{1}{2}$ in (17).

$$y(x) = MS\left(q,\lambda;x=\cos^2 z,\eta=\frac{1}{4}qx^2\right)$$

$$
\begin{aligned}
&= x^{\frac{1}{2}}\Bigg\{\sum_{i_0=0}^{\infty}\frac{1}{(\frac{5}{4})_{i_0}(1)_{i_0}}\eta^{i_0}\\
&+ \sum_{n=1}^{\infty}\Bigg\{\prod_{k=0}^{n-1}\Bigg\{\int_0^1 dt_{n-k}\; t_{n-k}^{-1+\frac{1}{2}(n-k)}\int_0^1 du_{n-k}\; u_{n-k}^{-\frac{3}{4}+\frac{1}{2}(n-k)}\\
&\times I_0\left(2\sqrt{w_{n+1-k,n}(1-t_{n-k})(1-u_{n-k})}\right)\\
&\times \left(w_{n-k,n}^{-\frac{1}{2}(n-k-\frac{1}{2})}\left(w_{n-k,n}\partial_{w_{n-k,n}}\right)^2 w_{n-k,n}^{\frac{1}{2}(n-k-\frac{1}{2})}-\frac{1}{4^2}(\lambda+2q)\right)\Bigg\}\\
&\times \sum_{i_0=0}^{\infty}\frac{1}{(\frac{5}{4})_{i_0}(1)_{i_0}}w_{1,n}^{i_0}\Bigg\}x^n\Bigg\} \qquad (29)
\end{aligned}
$$

replaced α and x by $\frac{1}{4}$ and $2\sqrt{\eta}$ in (14).

$$
\begin{aligned}
I_{\frac{1}{4}}\left(2\sqrt{\eta}\right) &= \frac{\eta^{\frac{1}{8}}}{\Gamma(\frac{5}{4})}\sum_{l=0}^{\infty}\frac{1}{l!\;(\frac{5}{4})_l}\eta^l\\
&= \frac{\eta^{\frac{1}{8}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}\int_{-1}^{1}dv_0\;(1-v_0^2)^{-\frac{1}{4}}\exp\left(-2\sqrt{\eta}\;v_0\right) \qquad (30)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{\frac{1}{4}}\left(2\sqrt{w_{1,n}}\right) &= \frac{w_{1,n}^{\frac{1}{8}}}{\Gamma(\frac{5}{4})}\sum_{l=0}^{\infty}\frac{1}{l!\;(\frac{5}{4})_l}w_{1,n}^l\\
&= \frac{w_{1,n}^{\frac{1}{8}}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}\int_{-1}^{1}dv_0\;(1-v_0^2)^{-\frac{1}{4}}\exp\left(-2\sqrt{w_{1,n}}\;v_0\right) \qquad (31)
\end{aligned}
$$

Substitute (30) and (31) into (29).

Remark 5. The integral representation of Mathieu equation of the second kind for an infinite series about $x = 0$ using 3TRF is given by

$$
\begin{aligned}
y(x) &= MS\left(q,\lambda;x=\cos^2 z,\eta=\frac{1}{4}qx^2\right)\\
&= \Gamma(5/4)x^{\frac{1}{2}}\Bigg\{\eta^{-\frac{1}{8}}I_{\frac{1}{4}}\left(2\sqrt{\eta}\right)\\
&+ \sum_{n=1}^{\infty}\Bigg\{\prod_{k=0}^{n-1}\Bigg\{\int_0^1 dt_{n-k}\; t_{n-k}^{-1+\frac{1}{2}(n-k)}\int_0^1 du_{n-k}\; u_{n-k}^{-\frac{3}{4}+\frac{1}{2}(n-k)}
\end{aligned}
$$

$$\times \quad I_0\left(2\sqrt{w_{n+1-k,n}(1-t_{n-k})(1-u_{n-k})}\right)$$
$$\times \quad \left(w_{n-k,n}^{-\frac{1}{2}(n-k-\frac{1}{2})}\left(w_{n-k,n}\partial_{w_{n-k,n}}\right)^2 w_{n-k,n}^{\frac{1}{2}(n-k-\frac{1}{2})} - \frac{1}{4^2}(\lambda+2q)\right)\Bigg\}$$
$$\times \quad w_{1,n}^{-\frac{1}{8}} I_{\frac{1}{4}}\left(2\sqrt{w_{1,n}}\right)\Bigg\}x^n\Bigg\} \tag{32}$$

As we see (28) and (32), modified Bessel function recurs in each of sub-integral forms. We can transform Mathieu functions from these integral forms to other well-known special functions such as Kummer function, Legendre function, hypergeometric function, Laguerre function etc.

4. Asymptotic behavior of the function $y(x)$ and the boundary condition for $x = \cos^2 z$ for infinite series

As $n \gg 1$ (for sufficiently large), (5a) and (5b) are

$$\lim_{n\gg 1} A_n = A = 1 \tag{33a}$$

And,

$$\lim_{n\gg 1} B_n = B = \frac{q}{n^2} \tag{33b}$$

As $n \gg 1$, (33b) is extremely smaller than (33a). Put (33a) with $B_n = 0$ into (4).

$$c_{n+1} = c_n \tag{34}$$

Plug (34) into the power series expansion where $\sum_{n=0}^{\infty} c_n x^n$, putting $c_0 = 1$ for simplicity.

$$\lim_{n\gg 1} y(x) = \frac{1}{1-x} \qquad \text{where } 0 \le x < 1 \tag{35}$$

For being convergent of $y(x)$ in (35), an independent variable $x = \cos^2 z$ should be less than 1. In general, in various areas of the modern physics, we require that a function $y(x)$ should be convergent at $x = 1$. Mathieu polynomial of type 2 merely makes possible that a $y(x)$ is convergent at $x = 1$, and its formal series solutions are available in chapter 7 of Ref.[16].

I show power series expansions in closed forms of Mathieu equation for an infinite series in this paper analytically. Also, I derive integral forms of Mathieu equation from its general solution in series. It is quiet important that a

modified Bessel function recurs in each of sub-integrals of the Mathieu function, because we can investigate how this function is associated with other well known special functions such as Bessel, Kummer, hypergeometric and Laguerre functions, etc. In chapter 7 [16], I derive local solutions of Mathieu equation for a polynomial of type 2 by using similar methods what I do in this paper: (slowromancapi@) power series expansions in closed forms, (slowromancapii@) integral representations and (slowromancapiii@) generating functions. And in future papers, I will construct orthogonal relations, normalized constants and expectation values of physical quantities from generating functions for Mathieu polynomials analytically.

5. Application

1. By using the methods on the above, we can apply power series expansions of Mathieu equation and their integral forms into various modern physics areas.[1, 3, 18, 20, 23, 26, 27, 29] For example, in general Mathieu equation arises from two-dimensional vibrational problems in elliptical coordinates with physical points of a view[31]. Its equation is derived from the Helmholtz equation in elliptic cylinder coordinates by using the method of separation of variables (see (5)-(7) in p.610 in Ref.[31]). Using power series expansions of Mathieu equation, it might be possible to obtain specific eigenvalues for the wave equation in vibrational systems. The normalized constants for wave functions and their expectation values for the entire region might be also possible to be constructed by applying generating functions for Mathieu polynomials of type 2.[2]

2. In "Examples of Heun and Mathieu functions as solutions of wave equations in curved spaces"[4], "Dirac equation in the background of the Nutku helicoid metric"[5], two authors consider the Dirac equation in the background of the Nutku helicoid metric in five dimensions. They obtain solutions for the four different components by using the Newman-Penrose formalism[24](see (8a)-(8d) Ref.[5]). And they separate uncoupled equations for the lower components into two ordinary differential equations: angular equation is of the Mathieu type and radial equation is of the double confluent form which can be reduced to the Mathieu equation (see (11), (13) Ref.[5]). Using power series expansions

[2]several authors treat local solutions of Mathieu equation as an polynomial. In chapter 7 [16], I construct Mathieu functions for a polynomial of type 2. Indeed, their generating functions are derived in mathematical rigour. In this paper I show analytic solutions of Mathieu equation for an infinite series

and their integral representation of Mathieu equation, it might be possible to obtain eigenvalues and normalized wave functions at various regions.[3]

6. Series "Special functions and three term recurrence formula (3TRF)"

This paper is 5th out of 10.

1. "Approximative solution of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system" [6]–in order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalizes all existing theories of confluent hypergeometric types. We call it the grand confluent hypergeometric (GCH) function. Our new solution produces previously unknown extra hidden quantum numbers relevant for the description of supersymmetry and generating new mass formulas.

2. "Generalization of the three-term recurrence formula and its applications" [7]–generalize the three term recurrence relation in a linear differential equation. Obtain general summation solutions in series of the three term recurrence for an infinite series and a polynomial of type 1.

3. "The analytic solution for the power series expansion of Heun function" [8]–by applying the 3TRF, power series expansions in closed forms of Heun equation are constructed for an infinite series and polynomials of type 1.

4. "Asymptotic behavior of Heun function and its integral formalism" [9]–by applying the 3TRF, integrals of Heun equation are derived for an infinite series and polynomials of type 1. And an asymptotic behavior of Heun equation is analyzed including the radius of convergence.

5. "The power series expansion of Mathieu function and its integral formalism" [10]–apply the 3TRF, and analyze the power series expansion of Mathieu equation for an infinite series and its integral form.

6. "Lame equation in the algebraic form" [11]–by applying the 3TRF, power series expansions of Lame equation in the algebraic form are analyzed for an infinite series and a polynomial of type 1 with their integral forms.

[3]In the future I will construct analytic solutions of the Mathieu polynomial and its eigenvalues of these two problems.

7. "Power series and integral forms of Lame equation in Weierstrass's form" [12]–by applying the 3TRF, power series expansions of Lame equation in Weierstrass's form are derived for an infinite series and a polynomial of type 1 including their representation in terms of integrals.

8. "The generating functions of Lame equation in Weierstrass's form" [13]–by applying the general integral of Lame equation in Weierstrass's form for a polynomial of type 1, generating functions of Lame polynomials are derived.

9. "Analytic solution for grand confluent hypergeometric function" [14]–apply the 3TRF, and formulate analytic solutions of the GCH functions for an infinite series and a polynomial of type 1. Replacing μ and $\varepsilon\omega$ by 1 and $-q$ transforms the GCH functions into the biconfluent Heun functions.

10. "The integral formalism and the generating function of grand confluent hypergeometric function" [15]–by applying the 3TRF, integrals of the GCH equation for an infinite series and a polynomial of type 1 are constructed including generating functions of the GCH polynomials.

Acknowledgment

I thank Bogdan Nicolescu. The discussions I had with him on number theory was of great joy.

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