EXISTENCE OF POSITIVE SOLUTION
FOR A NONLINEAR THREE-POINT
BOUNDARY VALUE PROBLEM

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Abstract: The existence of positive solution is obtained for the following nonlinear three-point boundary value problem

\[
\begin{aligned}
&u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1) \\
&\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta),
\end{aligned}
\]

where

\[
\beta, \gamma \geq 0, 0 < \eta < 1, 0 < \alpha \leq \frac{1}{\eta} \text{ and } d = \beta(1 - \alpha\eta) + \gamma(1 - \alpha) > 0.
\]

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1. Introduction

Recently, multi-point boundary value problems have received much attention from many authors [2], 2–5. In particular, Ma [5] obtained the existence of one positive solution for the boundary value problem

\[ u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \]

\[ u(0) = 0, \quad u(1) = \alpha u(\eta), \]

where \(0 < \eta < 1\) and \(0 < \alpha < 1/\eta\), and Liu [4] discussed the existence of positive solutions to the boundary value problem

\[ u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \]

\[ u'(0) = 0, \quad u(1) = \alpha u(\eta), \]

where \(0 < \eta, \alpha < 1\). In the above papers [5, 4], the authors require that \(f\) is either superlinear or sublinear, and their main results are based upon an application of the well-known Krasnoselskii’s Fixed Point Theorem [1].

The purpose of this paper is to study the following more general nonlinear three-point boundary value problem

\[ u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1) \]

\[ \beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta), \]

where \(\beta, \gamma \geq 0, 0 < \eta < 1, 0 < \alpha < \frac{1}{\eta}\) and \(d = \beta(1-\alpha\eta) + \gamma(1-\alpha) > 0\). In Section 3, we establish some existence results of positive solution for the problem (5) and (6), and we do not require that \(f\) is either superlinear or sublinear.

Throughout, we assume that the following conditions are satisfied:

(A1) \(f \in C([0,\infty),[0,\infty))\).

(A2) \(a \in C([0,1],[0,\infty))\) and there exists \(x_0 \in [\eta,1]\) such that \(a(x_0) > 0\).

(A3) \(d > 0\).
2. Preliminary Lemmas

Lemma 2.1. Let $d \neq 0$. Then for $h \in C[0,1]$, the problem

$$u''(t) + h(t) = 0, \quad t \in (0,1),$$

(7)

$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta),$$

(8)

has a unique solution

$$u(t) = -\int_0^t (t-s)h(s)ds$$

$$+ \frac{\beta t + \gamma}{d} \left[ \int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta-s)h(s)ds \right].$$

Proof. From (7) and (8), it is easy to see that the result holds. \qed

Lemma 2.2. Let $d > 0$. If $h \in C[0,1]$ and $h \geq 0$, then the unique solution $u$ of the problem (7) and (8) satisfies $u(t) \geq 0$ for $t \in [0,1]$.

Proof. From the fact that $u''(t) = -h(t) \leq 0$, we know that the graph of $u(t)$ is concave down on $(0,1)$. So it suffices to prove that $u(0) \geq 0$ and $u(1) \geq 0$.

To show that $u(0) \geq 0$, there are two cases to be considered. We first consider the case $0 < \alpha < 1$. In this case

$$u(0) = \frac{\gamma}{d} \left[ \int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta-s)h(s)ds \right]$$

$$\geq \frac{\gamma}{d} \left[ \int_0^1 (1-s)h(s)ds - \alpha \int_0^1 (1-s)h(s)ds \right]$$

$$= \frac{\gamma(1-\alpha)}{d} \int_0^1 (1-s)h(s)ds$$

$$\geq 0.$$
Next we consider the case $\alpha \geq 1$. In this case

$$u(0) = \frac{\gamma}{d} \left[ \int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds \right]$$

$$= \frac{\gamma}{d} \left[ \int_0^\eta [(1 - \alpha \eta) + (\alpha - 1)s] h(s)ds + \int_\eta^1 (1 - s)h(s)ds \right]$$

$$\geq 0.$$ 

Furthermore, we know that

$$u(1) = - \int_0^1 (1-s)h(s)ds$$

$$+ \frac{\beta + \gamma}{d} \left[ \int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds \right]$$

$$= \frac{\alpha(\beta \eta + \gamma)}{d} \int_0^1 (1-s)h(s)ds - \frac{\alpha(\beta + \gamma)}{d} \int_0^\eta (\eta - s)h(s)ds$$

$$\geq \frac{\alpha(\beta + \gamma)}{d} \int_0^\eta (\beta s + \gamma)h(s)ds \geq 0.$$ 

The proof is complete. $\square$

Similar to the proof of Lemma 4 of [5], we can obtain the following lemma.

**Lemma 2.3.** Let $d > 0$. If $h \in C[0,1]$ and $h \geq 0$, then the unique solution $u$ of the problem (7) and (8) satisfies

$$\min_{t \in [\eta, 1]} u(t) \geq \sigma \|u\|,$$

where

$$\sigma = \min \left\{ \frac{\alpha(1 - \eta)}{1 - \alpha \eta}, \alpha \eta, \eta \right\} \text{ and } \|u\| = \max_{t \in [0,1]} |u(t)|.$$
The following fixed point theorem (see [1]) is very crucial in our arguments.

**Theorem 2.4.** (Krasnoselskii’s Fixed Point Theorem) Let $E$ be a Banach space, and $K$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ are open subsets of $E$ with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let

$$\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\Phi u\| \leq \|u\|, \forall u \in K \cap \partial \Omega_1$ and $\|\Phi u\| \geq \|u\|, \forall u \in K \cap \partial \Omega_2,$

or

(ii) $\|\Phi u\| \geq \|u\|, \forall u \in K \cap \partial \Omega_1$ and $\|\Phi u\| \leq \|u\|, \forall u \in K \cap \partial \Omega_2.$

Then $\Phi$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

For convenience, we let

$$D = \frac{d}{\beta + \gamma} \left[ \int_0^1 (1 - s)a(s)ds \right]^{-1},$$

and

$$C = \frac{d}{\beta \eta + \gamma} \left[ \int_\eta^1 (1 - s)a(s)ds \right]^{-1}.$$

**Theorem 3.1.** Assume that there exist two different positive numbers $\nu$ and $\mu$ such that

$$f(u) \leq \nu D \text{ for } u \in [0, \nu] \quad \text{(9)}$$

and

$$f(u) \geq \mu C \text{ for } u \in [\sigma \mu, \mu]. \quad \text{(10)}$$
Then, the boundary value problem (5) and (6) has at least one positive solution \( u \) such that

\[
\min \{ \nu, \mu \} \leq \| u \| \leq \max \{ \nu, \mu \}.
\]

Proof. Set

\[
K = \left\{ u \mid u \in C[0, 1], u \geq 0, \min_{t \in [\eta, 1]} u(t) \geq \sigma \| u \| \right\}
\]

and

\[
\Phi u(t) = -\int_0^t (t - s)a(s)f(u(s))ds + \frac{\beta t + \gamma}{d} \left[ \int_0^1 (1 - s)a(s)f(u(s))ds - \alpha \int_0^\eta (\eta - s)a(s)f(u(s))ds \right].
\]

Then, it is obvious that \( K \) is a cone in \( C[0, 1] \). Moreover, by Lemma 2.3, \( \Phi(K) \subset K \). It is also easy to check that \( \Phi : K \to K \) is completely continuous and that \( u \) is a solution of the problem (5) and (6) if and only if \( u \) is a fixed point of \( \Phi \).

Without loss of generality, we may assume \( \nu < \mu \).

Set

\[
\Omega_1 = \{ u \in C[0, 1] \mid \| u \| < \nu \} \quad \text{and} \quad \Omega_2 = \{ u \in C[0, 1] \mid \| u \| < \mu \}.
\]

For \( u \in K \cap \partial \Omega_1 \), we have from (9) that

\[
\Phi u(t) = -\int_0^t (t - s)a(s)f(u(s))ds + \frac{\beta t + \gamma}{d} \left[ \int_0^1 (1 - s)a(s)f(u(s))ds - \alpha \int_0^\eta (\eta - s)a(s)f(u(s))ds \right] \\
\leq \frac{\beta t + \gamma}{d} \int_0^1 (1 - s)a(s)f(u(s))ds
\]
\[
\begin{align*}
\leq & \quad \nu D \frac{\beta + \gamma}{d} \int_0^1 (1 - s)a(s)ds \\
= & \quad \nu = \|u\|, \quad t \in [0, 1],
\end{align*}
\]
and so,

\[
\|\Phi u\| \leq \|u\| \text{ for } u \in K \cap \partial \Omega_1. \tag{11}
\]

For \( u \in K \cap \partial \Omega_2 \), we get from (10) that

\[
\Phi u(\eta) = -\int_0^\eta (\eta - s)a(s)f(u(s))ds \\
+ \frac{\beta \eta + \gamma}{d} \left[ \int_0^1 (1 - s)a(s)f(u(s))ds \\
- \alpha \int_0^\eta (\eta - s)a(s)f(u(s))ds \right] \\
\geq \frac{\beta \eta + \gamma}{d} \int_\eta^1 (1 - s)a(s)f(u(s))ds \\
\geq \mu C \frac{\beta \eta + \gamma}{d} \int_\eta^1 (1 - s)a(s)ds \\
= \mu = \|u\|.
\]

So,

\[
\|\Phi u\| \geq \|u\| \text{ for } u \in K \cap \partial \Omega_2. \tag{12}
\]

Therefore, it follows from (11), (12) and Theorem 2.4 that the boundary value problem (5) and (6) has at least one positive solution \( u \) such that

\[
\nu \leq \|u\| \leq \mu.
\]

The proof is complete.
Corollary 3.1. The boundary value problem (5) and (6) has at least one positive solution if either
\[
\lim_{u \to 0} \frac{f(u)}{u} < D \text{ and } \lim_{u \to \infty} \frac{f(u)}{u} > \frac{C}{\sigma}, \tag{13}
\]
or
\[
\lim_{u \to 0} \frac{f(u)}{u} > \frac{C}{\sigma} \text{ and } \lim_{u \to \infty} \frac{f(u)}{u} < D. \tag{14}
\]

Proof. First, we suppose (13) holds. Since \(\lim_{u \to 0} \frac{f(u)}{u} < D\), we may choose \(\nu > 0\) such that
\[
f(u) \leq Du \leq D\nu \text{ for } u \in [0, \nu]. \tag{15}
\]
Furthermore, since \(\lim_{u \to \infty} \frac{f(u)}{u} > \frac{C}{\sigma}\), there exists \(d > 0\) such that
\[
f(u) \geq \frac{C}{\sigma} u \text{ for } u \geq d. \tag{16}
\]
Let \(\mu > \max\{\nu, \frac{d}{\sigma}\}\), so
\[
f(u) \geq \frac{C}{\sigma} u \geq C\mu \text{ for } u \in [\sigma\mu, \mu]. \tag{17}
\]
Therefore, by (15), (16) and Theorem 3.1, it follows that the boundary value problem (5) and (6) has at least one positive solution.

Next, we suppose (14) holds. In view of \(\lim_{u \to 0} \frac{f(u)}{u} > \frac{C}{\sigma}\), we may choose \(\mu > 0\) such that
\[
f(u) \geq \frac{C}{\sigma} u \text{ for } u \in (0, \mu]. \tag{18}
\]
Especially, for \(u \in [\sigma\mu, \mu]\), we have
\[
f(u) \geq \frac{C}{\sigma} u \geq C\mu. \tag{19}
\]
On the other hand, since \(\lim_{u \to \infty} \frac{f(u)}{u} < D\), there exists \(d > 0\) such that
\[
f(u) \leq Du \text{ for } u \geq d. \tag{20}
\]
We consider two cases:
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Case (i). $\lim_{u \to \infty} f(u) = +\infty$. In this case, there exist $c_n (n = 1, 2, \cdots) \to +\infty$ so that

$$f(c_n) = \max_{0 \leq u \leq c_n} f(u).$$

(19)

Choose $n_0$ such that $\nu = c_{n_0} > \max \{\mu, d\}$, then by (18) and (19), we have

$$f(u) \leq f(\nu) \leq D\nu \text{ for } u \in [0, \nu].$$

(20)

Case (ii). If $\lim_{u \to \infty} f(u) < +\infty$, then $f$ is bounded on $[0, +\infty)$. Let

$$M = \sup \{f(u) \mid u \in [0, +\infty)\},$$

and choose $\nu > \max \{\mu, \frac{M}{D}\}$, then we have

$$f(u) \leq M \leq D\nu \text{ for } u \in [0, +\infty).$$

Especially, we have

$$f(u) \leq D\nu \text{ for } u \in [0, \nu].$$

(21)

Therefore, we may get $\nu > \mu > 0$ such that

$$f(u) \leq D\nu \text{ for } u \in [0, \nu].$$

(22)

Then, by (17), (22) and Theorem 3.1, it follows that the boundary value problem (5) and (6) has at least one positive solution. The proof is complete.

\[ \square \]

References


