INFINITELY MANY POSITIVE SOLUTIONS
OF A NONLINEAR THREE-POINT
BOUNDARY VALUE PROBLEM

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Abstract: In this paper, existence criteria for infinitely many positive
solutions of the nonlinear three-point boundary value problem
\[
\begin{align*}
& u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \quad t \in (0, 1), \\
& u(0) = 0, \quad u(1) = \alpha u(\eta)
\end{align*}
\]
are established by using the Krasnosel’skii’s Fixed Point Theorem for
operators on cone.

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1. Introduction

Recently, the study of the existence of positive solutions for multi-point
boundary value problems has evolved rapidly [3], 2–8. In particular, Ma
and Wang [9], and Sun et al [10] considered the existence of one or three
positive solutions for more general three-point boundary value problem
\[
\begin{align*}
& u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \quad t \in (0, 1)
\end{align*}
\]
under the suitable conditions, where:

(H1) \( f \in C([0, \infty), [0, \infty)) \).

(H2) \( h \in C([0, 1], [0, \infty)) \) and there exists \( x_0 \in [0, 1] \) such that \( h(x_0) > 0 \).

(H3) \( a \in C[0, 1], b \in C([0, 1], (-\infty, 0)) \).

(H4) \( 0 < \alpha \phi_1(\eta) < 1 \), here \( \phi_1 \) is the unique solution of the linear boundary value problem

\[
\phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0, \quad t \in (0, 1),
\]

\[
\phi_1(0) = 0, \quad \phi_1(1) = 1.
\]

Their main tool is Krasnosel’skii’s Fixed Point Theorem (see [2, 1]) or Leggett-Williams’ Fixed Point Theorem (see [6]).

In this paper we will continue to consider the existence of infinitely many positive solutions for the above boundary value problem and our main tool is the following well-known fixed point theorem, which is due to Krasnosel’skii and Guo.

**Theorem 1.1.** Let \( E \) be a Banach space, and \( P \) be a cone in \( E \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2 \), and let \( A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P \) be a completely continuous operator such that either:

(i) \( \|Au\| \leq \|u\|, \forall u \in P \cap \partial \Omega_1 \) and \( \|Au\| \geq \|u\|, \forall u \in P \cap \partial \Omega_2 \),

or

(ii) \( \|Au\| \geq \|u\|, \forall u \in P \cap \partial \Omega_1 \) and \( \|Au\| \leq \|u\|, \forall u \in P \cap \partial \Omega_2 \).

Then \( A \) has a fixed point in \( P \cap (\overline{\Omega}_2 \setminus \Omega_1) \).
2. Main Results

To state the main result of this paper, we need the following lemma, which was established by Ma and Wang, see [9].

**Lemma 2.1.** Assume that (H3) holds. Let \( \phi_1 \) and \( \phi_2 \) be the solutions of equation (1.3) and

\[
\begin{aligned}
\phi_2''(t) + a(t)\phi_2'(t) + b(t)\phi_2(t) &= 0, \quad t \in (0,1), \\
\phi_2(0) &= 1, \quad \phi_2(1) = 0.
\end{aligned}
\]  

(2.1)

Then:

(i) \( \phi_1 \) is strictly increasing on \([0,1]\);

(ii) \( \phi_2 \) is strictly decreasing on \([0,1]\).

In view of \( (H2) \), \( h \in C([0,1], [0, \infty)) \) and there exists \( x_0 \in [0,1] \) such that \( h(x_0) > 0 \), and hence we may assume that \( x_0 \in (0,1) \). Take \( \delta \in (0, \frac{1}{2}) \) such that \( x_0 \in (\delta, 1 - \delta) \).

For convenience, we let

\[
G(t, s) = \frac{1}{\phi_1(0)} \begin{cases} 
\phi_1(t)\phi_2(s), & s \geq t, \\
\phi_1(s)\phi_2(t), & s \leq t,
\end{cases}
\]

\[
D = \left[ \max_{t \in [0,1]} \int_0^1 G(t, s)p(s)h(s)ds + \frac{\alpha}{1 - \alpha \phi_1(\eta)} \int_0^1 G(\eta, s)p(s)h(s)ds \right]^{-1},
\]

and

\[
C = [1 - \alpha \phi_1(\eta)] \left[ \int_{\delta}^{1-\delta} G(\eta, s)p(s)h(s)ds \right]^{-1},
\]

where \( p(t) = \exp \left( \int_0^t a(s)ds \right) \).

For the function \( G(t, s) \), it follows from Lemma 2.1 that

\[
0 \leq G(t, s) \leq G(s, s), \quad (t, s) \in [0,1] \times [0,1]
\]  

(2.2)
and
\[ G(t,s) \geq \gamma G(s,s), \quad (t,s) \in [\delta, 1-\delta] \times [0,1], \quad (2.3) \]
where \( \gamma = \min \{ \phi_1(\delta), \phi_2(1-\delta) \} \).

Our main result is the following theorem.

**Theorem 2.1.** Assume that (H1)-(H4) hold and that there exist two positive sequences \( \{a_k\}_{k=1}^{\infty} \) and \( \{b_k\}_{k=1}^{\infty} \) such that
\[ a_{k+1} < b_k < a_k, \quad k = 1, 2, \cdots. \]
Then the boundary value problem (1.1) and (1.2) has infinitely many positive solutions if the following conditions hold:
\[ f(u) \leq Da_k, \quad u \in [0, a_k], \quad k = 1, 2, \cdots, \quad (2.4) \]
and
\[ f(u) \geq Cb_k, \quad u \in [\gamma b_k, b_k], \quad k = 1, 2, \cdots. \quad (2.5) \]

**Proof.** Let \( E \) be the set \( C[0,1] \) of all real continuous functions defined on \([0,1]\) endowed with the usual linear structure and the maximum norm. Set
\[ P = \left\{ u \in E : u(t) \geq 0, t \in [0,1], \min_{t \in [\delta, 1-\delta]} u(t) \geq \gamma \| u \| \right\}. \]
Then it is easily seen that \( P \) is a cone in \( E \). For \( u \in P \), define
\[ (Au)(t) = \int_0^1 G(t,s)p(s)h(s)f(u(s))ds + \frac{\alpha \phi_1(t)}{1-\alpha \phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f(u(s))ds, \quad t \in [0,1]. \]
It is easy to check that [9] \( A : P \to P \) is completely continuous and fixed points of \( A \) are solutions of the boundary value problem (1.1) and (1.2). Let
\[ \Omega_{a_k} = \{ u \in E : \| u \| < a_k \} \quad \text{and} \quad \Omega_{b_k} = \{ u \in E : \| u \| < b_k \}. \]
Fix $k$ and let $u \in P \cap \partial \Omega_{a_k}$, then for $s \in [0, 1]$, we have
\[ 0 \leq u(s) \leq \|u\| = a_k. \tag{2.6} \]
It follows from (2.6) and (2.4) that
\[
(Au)(t) = \int_0^1 G(t, s)p(s)h(s)f(u(s))ds \\
+ \frac{\alpha \phi_1(t)}{1 - \alpha \phi_1(\eta)} \int_0^1 G(\eta, s)p(s)h(s)f(u(s))ds \\
\leq Da_k \left[ \int_0^1 G(t, s)p(s)h(s)ds \\
+ \frac{\alpha \phi_1(t)}{1 - \alpha \phi_1(\eta)} \int_0^1 G(\eta, s)p(s)h(s)ds \right] \\
\leq Da_k \max_{t \in [0,1]} \left[ \int_0^1 G(t, s)p(s)h(s)ds \\
+ \frac{\alpha \phi_1(t)}{1 - \alpha \phi_1(\eta)} \int_0^1 G(\eta, s)p(s)h(s)ds \right] \\
\leq Da_k \left[ \max_{t \in [0,1]} \int_0^1 G(t, s)p(s)h(s)ds \\
+ \frac{\alpha}{1 - \alpha \phi_1(\eta)} \int_0^1 G(\eta, s)p(s)h(s)ds \right] \\
= a_k = \|u\|, \quad t \in [0, 1].
\]
So,
\[ \|Au\| \leq \|u\|, \quad u \in P \cap \partial \Omega_{a_k}. \tag{2.7} \]
Now let $u \in P \cap \partial \Omega_{b_k}$, then for $s \in [\delta, 1 - \delta]$, we have
\[ b_k = \|u\| \geq u(s) \geq \min_{s \in [\delta, 1 - \delta]} u(s) \geq \gamma \|u\| = \gamma b_k. \tag{2.8} \]
It follows from (2.8) and (2.5) that

\[(Au)(\eta) = \frac{1}{1 - \alpha \phi_1(\eta)} \int_0^1 G(\eta, s)p(s)h(s)f(u(s))ds\]

\[\geq \frac{1}{1 - \alpha \phi_1(\eta)} \int_{1-\delta}^1 G(\eta, s)p(s)h(s)f(u(s))ds\]

\[\geq \frac{Cb_k}{1 - \alpha \phi_1(\eta)} \int_{1-\delta}^1 G(\eta, s)p(s)h(s)ds\]

\[= b_k = \|u\| .\]

And so,

\[\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_{b_k}. \quad (2.9)\]

By (2.7), (2.9) and Theorem 1.1 the operator \(A\) has a fixed point \(u_k \in P \cap \partial\Omega_{b_k}\). Since \(k\) is arbitrary, the proof is complete. \(\square\)

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References


