CONVERGENCE OF THE SBA ALGORITHM APPLYING TO SOLVE VOLterra NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND

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Abstract: In this paper we show the convergence of the SBA method on the Volterra nonlinear integro-differential equations of second kind.

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1. Introduction

In this paper, we show the convergence of the SBA method (combination of the principle of Picard, Adomian method and the successive approximations) on the nonlinear integro-differential equations of Volterra second kind under the
form:
\[
\begin{align*}
\varphi'(x) &= f(x) + \lambda \int_a^x K(x,t)g(\varphi(t)) \, dt \\
\varphi(a) &= c
\end{align*}
\]
where \( g \) is such as \( g(\varphi(t)) = l(\varphi(t)) + N(\varphi(t)), \lambda > 0, l(\varphi(t)) = \varphi(t), \)
a \( \leq t \leq x \leq T < +\infty \) and \( N \) is nonlinear.

2. Convergence of the SBA Method

**Theorem 1.** Let us consider the following Volterra integro-differential second kind

\[
(p) : \begin{cases}
\varphi'(x) = f(x) + \lambda \int_a^x K(x,t)g(\varphi(t)) \, dt \\
\varphi(a) = c
\end{cases}
\]

where \( g(\varphi(t)) = R(t) + N(\varphi(t)), \lambda > 0, \varphi \in C^1(\Sigma), f \in \Sigma \) and \( K \in C(\Sigma \times \Omega) \)

\((p)\) is equivalent to

\[
\varphi(x) = c + \int_a^x f(z) \, dz + \lambda \int_a^x \int_a^z K(z,t) R(t) \, dt \, dz + \lambda \int_a^x \int_a^z K(z,t) N(\varphi(t)) \, dt \, dz
\]

Applying the SBA method to \((E)\), we get:

\[
(p_{\text{app}}) \begin{cases}
\varphi_0^k(x) = c + \int_a^x f(z) \, dz + \lambda \int_a^x \int_a^z K(z,t) N(\varphi^{k-1}(t)) \, dt \, dz, k \geq 1 \\
\varphi_{n+1}^k(x) = \lambda \int_a^x \int_a^z K(z,t) \varphi_n^k(t) \, dt \, dz; n \geq 0
\end{cases}
\]

We suppose the following hypotheses:

\(H_1: \forall x \in \Sigma, |f(x)| \leq \beta, |c| = \mu\)

\(H_2 : \forall (x,t) \in \Sigma \times \Omega, |K(x,t)| \leq M\)

\(H_3 : \exists \varphi^0 \in H \) such as \( N(\varphi^0(t)) = 0 \) and for \( k = p, N(\varphi^p(t)) = 0 \).

If the SBA algorithm associated to \((E_{\text{app}})\) converges to the step \( k = 1 \), then the solution \( \varphi(x) \) of the equation \((E)\) is unique, such as \( \varphi(x) = \lim_{k \to +\infty} \varphi^k(x) \).
Proof. At the step $k = 1$, we have:

$$(p_{app}) : \begin{align*}
\varphi_0^1 (x) &= c + \int_a^x f(z) \, dz \\
\varphi_{n+1}^1 (x) &= \lambda \int_a^x \int_a^z K(z,t) \varphi_n^1 (t) \, dt dz; \, n \geq 0
\end{align*}$$ (4)

Applying $H_1$ and $H_2$

$$\Rightarrow (p_{app}) : \begin{align*}
|\varphi_0^1 (x)| &\leq |c| + \int_a^x |f(z)| \, dz \leq \mu + \beta (x-a) \\
|\varphi_{n+1}^1 (x)| &\leq \lambda \int_a^x \int_a^z |K(z,t)| |\varphi_n^1 (t)| \, dt dz; \, n \geq 0
\end{align*}$$ (5)

then

$$\begin{align*}
|\varphi_0^1 (x)| &\leq \mu + \beta (x-a) \\
|\varphi_1^1 (x)| &\leq \lambda M \left[ \frac{\mu (x-a)^2}{2!} + \frac{\beta (x-a)^3}{3!} \right] \\
|\varphi_2^1 (x)| &\leq \lambda^2 M^2 \left[ \frac{\mu (x-a)^4}{4!} + \frac{\beta (x-a)^5}{5!} \right] \\
&\quad \ldots \\
|\varphi_n^1 (x)| &\leq \lambda^n M^n \left[ \frac{\mu (x-a)^{2n}}{(2n)!} + \frac{\beta (x-a)^{2n+1}}{(2n+1)!} \right]
\end{align*}$$ (6)

$$\psi_1^p (x) = \mu \sum_{n=0}^p \left[ \frac{\sqrt{\lambda M} (x-a)}{(2n)!} \right]^{2n} + \frac{\beta}{\sqrt{\lambda M}} \sum_{n=0}^p \left[ \frac{\sqrt{\lambda M} (x-a)}{(2n+1)!} \right]^{2n+1}$$

so we obtain

$$\lim_{p \to +\infty} \psi_1^p (x) = \mu ch \left( \sqrt{\lambda M} (x-a) \right) + \gamma sh \left( \sqrt{\lambda M} (x-a) \right)$$ (7)

where $\gamma = \frac{\beta}{\sqrt{\lambda M}}$

which proved that $\left( \sum_{n=0}^{+\infty} \varphi_n^1 (x) \right)$ is absolutely convergent.

We supposed at the step $k = p \geq 1$, we have $N (\varphi^p (x)) = 0$ and we get at the step $k = p + 1$: 
\[
\begin{align*}
\left| \varphi_0^{p+1}(x) \right| & \leq \mu + \beta (x - a) \\
\left| \varphi_1^{p+1}(x) \right| & \leq \lambda M \left[ \frac{\mu (x - a)^2}{2!} + \frac{\beta (x - a)^3}{3!} \right] \\
\left| \varphi_2^{p+1}(x) \right| & \leq \lambda^2 M^2 \left[ \frac{\mu (x - a)^4}{4!} + \frac{\beta (x - a)^5}{5!} \right] \\
\vdots \\
\left| \varphi_n^{p+1}(x) \right| & \leq \lambda^n M^n \left[ \frac{\mu (x - a)^{2n}}{(2n)!} + \frac{\beta (x - a)^{2n+1}}{(2n+1)!} \right]
\end{align*}
\]  

\(\Rightarrow\)

\[
\sum_{n=0}^{+\infty} \left| \varphi_n^{p+1}(x) \right| \leq \mu \sum_{n=0}^{+\infty} \frac{\sqrt{\lambda M} (x - a)^{2n}}{(2n)!} + \gamma \sum_{n=0}^{+\infty} \frac{\sqrt{\lambda M} (x - a)^{2n+1}}{(2n+1)!}
\]

\(\Rightarrow\)

\[
\sum_{n=0}^{+\infty} \left| \varphi_n^{p+1}(x) \right| \leq \mu ch \left( \sqrt{\lambda M} (x - a) \right) + \gamma sh \left( \sqrt{\lambda M} (x - a) \right).
\]  

(9)

which proved that \(\left( \sum_{n=0}^{+\infty} \varphi_n^{p+1}(x) \right)\) is absolutely convergent, then we get

\[\varphi(x) = \lim_{k \to +\infty} \varphi^k(x).\]

Now, let us suppose that the equation (E) admits two distinct solutions \(\varphi(x)\) and \(\phi(x)\).

Taking \(\delta(x) = \varphi(x) - \phi(x)\), applying the SBA algorithm with the preceding hypotheses, we have:

\[
\begin{align*}
\delta_0^k(x) &= 0 \quad ; \quad k = 1, 2, \ldots \\
\delta_n^k(x) &= \lambda \int_a^x K(x,t) \delta_{n-1}^k(t) \ dt \quad ; \quad n = 1, 2, \ldots
\end{align*}
\]

(10)

which solution at each step \(k\) is \(\delta^k(x) = 0\). Then we get:

\[\delta(x) = \lim_{k \to +\infty} \delta^k(x) = 0.\]

Hence

\[\forall t \in [a, x], \delta(t) = \varphi(t) - \phi(t) = 0 \Rightarrow \varphi(t) = \phi(t)\]

(11)

which is opposed to our hypothesis, so the solution of the equation (E) is unique.
3. Illustrative Examples

In this section, we solve some examples of linear and nonlinear integral differential equation Volterra second kind.

3.1. Example 1

Let us consider the following nonlinear integral differential equation second kind of Volterra, which is the canonical form of Adomian:

\[
(E) : \begin{cases}
\frac{d\varphi(x)}{dx} = 1 + x (\ln (2) - 1) + \int_0^x \frac{\varphi(t)}{x+t} dt + \int_0^x \frac{\varphi^5(t) - t^2 \varphi^3(t)}{x+t} dt \\
\varphi(0) = 0
\end{cases}
\]

where \( \varphi \in C^1([0; T]) \) and \( 0 \leq t < x \leq T < +\infty \).

We obtain:

\[
\varphi(x) = x + (\ln (2) - 1) \frac{x^2}{2!} + \int_0^x \int_0^t \varphi(t) dt dz + \int_0^x \int_0^t \frac{\varphi^5(t) - t^2 \varphi^3(t)}{z+t} dt dz
\]

The modified SBA algorithm for this equation is the following:

\[
\begin{cases}
\varphi_0^{k}(x) = x + \int_0^x \int_0^t \frac{N(\varphi^{k-1}(t))}{z+t} dt dz ; k \geq 1 \\
\varphi_{1}(x) = (\ln (2) - 1) \frac{x^2}{2!} + \int_0^x \int_0^t \varphi_0^{k}(t) dt dz \\
\varphi_{n}(x) = \int_0^x \int_0^t \frac{\varphi_{n-1}(t)}{z+t} dt dz ; n \geq 2
\end{cases}
\]

where \( N(\varphi(t)) = \varphi^5(t) - t^2 \varphi^3(t) \).

At the step \( k = 1 \), applying the principle of Picard, for \( \varphi^0(x) = 0 \), we have \( N(\varphi^0(x)) = 0 \) and we calculate: \( \varphi_0^{1}(x) , \varphi_{1}^{1}(x) , ..., \varphi_{n}^{1}(x) \).
converges to \( \varphi^1(x) = \sum_{n=0}^{+\infty} \varphi^1_n(x) \Rightarrow \varphi^1(x) = x \).

At the step \( k = 2 \), we have \( N(\varphi^1(x)) = 0 \) and we calculate: \( \varphi^2_0(x), \varphi^2_1(x), ..., \varphi^2_n(x) \).

\[
\begin{align*}
\varphi^2_0(x) &= x \\
\varphi^2_1(x) &= 0 \\
&\vdots \\
\varphi^2_n(x) &= 0; \quad n \geq 1
\end{align*}
\]

converges to \( \varphi^2(x) = \sum_{n=0}^{+\infty} \varphi^2_n(x) \Rightarrow \varphi^2(x) = x \).

In the recursive way, we obtain: \( \varphi^1(x) = \varphi^2(x) = ... = \varphi^k(x) = x \). Therefore, we obtain the exact solution of equation \((E)\):

\[
\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = x. \quad (13)
\]

3.2. Example 2

Let us consider the following nonlinear integral equation second kind of Volterra, which is the canonical form of Adomian:

\[
(F) : \begin{cases} 
\frac{d\varphi(x)}{dx} = e^{x} (1 - xe^{x}) + \int_{0}^{x} e^{2x-t} \varphi(t) \, dt + \\
\int_{0}^{x} e^{2x-t} \left( \sqrt{e^{-2t}} \varphi^3(t) - \varphi^2(t) \right) \, dt \\
\varphi(0) = 1
\end{cases} \quad (14)
\]

where \( \varphi \in C^1([0; T]) \) and \( 0 \leq t < x \leq T < +\infty \).

We obtain:

\[
\varphi(x) = e^{x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} - \frac{1}{4} + \int_{0}^{x} \int_{0}^{x} e^{2z-x} \varphi(t) \, dt \, dz + \int_{0}^{x} \int_{0}^{x} e^{2z-t} N(\varphi(t)) \, dt \, dz
\]

Where \( N(\varphi(t)) = \sqrt{e^{-2t}} \varphi^3(t) - \varphi^2(t) \).

The modified SBA algorithm for this equation is the following:

\[
\begin{align*}
\varphi^k_0(x) &= e^{x} + \int_{0}^{x} \int_{0}^{x} e^{2z-t} N(\varphi^{k-1}(t)) \, dt \, dz \quad ; \quad k \geq 1 \\
\varphi^k_1(x) &= -\frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} - \frac{1}{4} + \int_{0}^{x} \int_{0}^{x} e^{2z-t} \varphi^k_0(t) \, dt \, dz \\
\varphi^k_n(x) &= \int_{0}^{x} \frac{\varphi^{k-1}(t)}{x + t} \, dt \quad ; \quad n \geq 2
\end{align*}
\]
At the step $k = 1$, applying the principle of Picard, for $\varphi^0 (x) = 0$, we have $N (\varphi^0 (x)) = 0$ and calculate $\varphi^1_0 (x), \varphi^1_1 (x), ..., \varphi^1_n (x)$.

\[
\begin{align*}
\varphi^1_0 (x) &= e^x \\
\varphi^1_1 (x) &= 0 \\
&\quad ...
\varphi^1_n (x) = 0; n \geq 1
\end{align*}
\]

converges to $\varphi^1 (x) = \sum_{n=0}^{+\infty} \varphi^1_n (x) \Rightarrow \varphi^1 (x) = e^x$.

At the step $k = 2$, we have $N (\varphi^1 (x)) = 0$ we calculate: $\varphi^2_0 (x), \varphi^2_1 (x), ..., \varphi^2_n (x)$.

\[
\begin{align*}
\varphi^2_0 (x) &= e^x \\
\varphi^2_1 (x) &= 0 \\
&\quad ...
\varphi^2_n (x) = 0; n \geq 1
\end{align*}
\]

converges to $\varphi^2 (x) = \sum_{n=0}^{+\infty} \varphi^2_n (x) \Rightarrow \varphi^2 (x) = e^x$.

In the recursive way, we get: $\varphi^1 (x) = \varphi^2 (x) = ... = \varphi^k (x) = e^x$. Therefore, we obtain the exact solution of equation (F)

\[
\varphi (x) = \lim_{k \to +\infty} \varphi^k (x) = e^x.
\]

(15)

3.3. Example 3

Let us consider the following nonlinear integral equation second kind of Volterra, which is the canonical form of Adomian:

\[
(H): \quad \begin{cases}
\frac{d \varphi (x)}{dx} = 1 - 2\sqrt{x} - \frac{2}{3} x \sqrt{x} + \int_0^x \frac{\varphi (t)}{\sqrt{x-t}} dt + \\
\int_0^x \frac{x (t \varphi (t)^2 - \varphi^4 (t))}{\sqrt{x-t}} dt \\
\varphi (0) = 0
\end{cases}
\]

(16)

where $\varphi \in C^1 ([0; T])$ and $0 \leq t < x \leq T < +\infty$.

We obtain:

\[
\varphi (x) = x - \frac{8}{15} x^2 \sqrt{x} + \int_0^x \int_0^z \frac{\varphi (t)}{\sqrt{z-t}} dt dz + \int_0^x \int_0^z \frac{(t^2 \varphi^2 (t) - \varphi^4 (t))}{\sqrt{z-t}} dt dz,
\]
Here \( N(\varphi(t)) = (t\varphi(t))^2 - \varphi^4(t) \).

The modified SBA algorithm for this equation is the following:

\[
\begin{cases}
\varphi^k_0(x) = x + \int_0^x \int_0^z \frac{N(\varphi^{k-1}_0(t))}{\sqrt{z-t}} dt dz; k \geq 1 \\
\varphi^k_1(x) = -\frac{8}{15}x^2\sqrt{x} + \int_0^x \int_0^z \frac{\varphi^k_0(t)}{\sqrt{z-t}} dt dz \\
\varphi^k_n(x) = \int_0^x \frac{\varphi^k_{n-1}(t)}{\sqrt{x-t}} dt; n \geq 2
\end{cases}
\]

At the step \( k = 1 \), applying the principle of Picard, for \( \varphi^0(x) = 0 \), we have \( N(\varphi^0(x)) = 0 \) and calculate \( \varphi^1_0(x), \varphi^1_1(x), ..., \varphi^1_n(x) \).

\[
\begin{cases}
\varphi^1_0(x) = x \\
\varphi^1_1(x) = 0 \\
... \\
\varphi^1_n(x) = 0; n \geq 1
\end{cases}
\]

converges to \( \varphi^1(x) = \sum_{n=0}^{+\infty} \varphi^1_n(x) \Rightarrow \varphi^1(x) = x \).

At the step \( k = 2 \), we have \( N(\varphi^1(x)) = 0 \) we calculate: \( \varphi^2_0(x), \varphi^2_1(x), ..., \varphi^2_n(x) \).

\[
\begin{cases}
\varphi^2_0(x) = x \\
\varphi^2_1(x) = 0 \\
\varphi^2_2(x) = 0 \\
... \\
\varphi^2_n(x) = 0; n \geq 1
\end{cases}
\]

converges to \( \varphi^2(x) = \sum_{n=0}^{+\infty} \varphi^2_n(x) \Rightarrow \varphi^2(x) = x \).

In the recursive way, we get: \( \varphi^1(x) = \varphi^2(x) = ... = \varphi^k(x) = x \). Therefore, we obtain the exact solution of equation (H)

\[
\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = x. \quad (17)
\]

4. Conclusion

In this paper, we proved the convergence of the algorithm SBA for the nonlinear integral equation Volterra of second kind and applied this convergence to solve successfully this kind of integral equations.
References


