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AN EFFICIENT ONE-EIGHT STEP HYBRID BLOCK METHOD FOR SOLVING SECOND ORDER INITIAL VALUE PROBLEMS OF ODES

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Abstract: In this article, an efficient hybrid block method for solving second order initial value problems of ODEs was proposed. The combination of power series and exponential function were used as an approximate solution for the derivation of the methods in modified block mode. The new method is derived via interpolation and collocation approaches and the proposed method was analyzed based on the characteristics of linear multistep method and the method was found to be zero-stable, consistent, convergent and the region of absolute stability of the proposed methods of one-eight step is plotted in the figures 1. The new proposed method gave an approximate solution to two real-life problems namely: simple harmonic motion and critically damped motion problems. Six numerical examples were solved to determine the efficiency and accuracy of the new method and the numerical solutions obtained yielded better results when compared with some existing methods in the literature.

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1. Introduction

In this manuscript, we present the approximate solution to second order initial value problems of the form:

$$y'' = f(x, y, y'), \quad y(x_0) = p, \quad y'(x_0) = q, \tag{1}$$

where f is the continuous function and fulfills the Lipchitz's condition that verifies the uniqueness and existence of a solution.

Some real world problems in Economics, applied Chemistry, Physics and Engineering have been modeled mathematically by second order ordinary differential equations (ODEs), for examples, dynamic problem, simple harmonic motion problem and critically damped motion problem are modeled using second Newton's law of motion and the result obtained lead to a system of second order differential equations. Majority of these models in ODEs are not always have theoretical solutions, thus approximate solutions and numerical approaches are often employed to solve many of these problems. Notable researchers like [1], [2] and [3] have used method of reduction of higher order ODEs into system of first order ODEs to give solutions to equation (1). The disadvantages of this method was reported very clearly by [4] and [5].

In order to cater for burden associated with reduction methods, several scholars like [6-10], adopted the techniques of predictor-corrector approach for solving initial value problems of ODEs. Although, this approach is good, because it gives direct solution to equation (1), but the approach is not easy to implement, since separate predictors or starting value are needed to implement the corrector and this reduced the accuracy of the method. The predictor-corrector is costly to examine, its computer subroutines are very complex to write and leads to longer CPU time. The disadvantages of the predictor-corrector methods was stated by [8].

In order to stay away from the limitation of the predictor-corrector methods, notable intellectuals such as [11-18] developed block methods in which approximations are simultaneously produced at distinct grid points in the interval of the integration. Block method is not so much costly in terms of the number evaluations compared to the linear multisteps and the major advantage of this method over traditional predictor-corrector method is that it gives better numerical solution when solving many problems in form of equation (1) directly.

The main aim of this paper is to review some existing methods for the numerical solution of the higher order initial value problems of ODEs and related problems, study their merits and demerits, and derive an efficient new hybrid block method for solving problem (1) and some related problems directly.

2. Mathematical Formulation of the Proposed Method

In this section, we use a combination of power series and exponential function as an approximate solution to be of the form:

$$Y(x) = \sum_{j=0}^{i+c-1} a_j x^j + a_{i+c} \sum_{j=0}^{i+c} \frac{\alpha^j x^j}{j!}, \tag{2}$$

where c represents the number of collocation points and i is the number of interpolation points. x^j represents polynomial basis function and $a_j's \in \Re$ are coefficients to be determined.

By substituting the second derivative of equation (2) into equation (1) we get:

$$Y''(x) = \sum_{j=0}^{i+c-1} a_j j(j-1)x^{j-2} + a_{i+c} \sum_{j=0}^{i+c} \frac{\alpha^j x^{j-2}}{(j-2)!} = f(x,y,y'). \tag{3}$$

Equations (2) and (3) are interpolated and collocated at the points $x_{n+i}, i = 0, \frac{1}{24}$ and $x_{n+i}, i = 0, \frac{1}{24}, \frac{1}{12}$, and $\frac{1}{8}$ to get a system of non linear equation of the form

$$AU = B, \tag{4}$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5]^T, \quad B = \left[y_n, y_{n+\frac{1}{24}}, f_n, f_{n+\frac{1}{24}}, f_{n+\frac{1}{12}}, f_{n+\frac{1}{8}}\right]^T,$$

and

$$U = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+\frac{1}{24}} & x^2_{n+\frac{1}{24}} & x^3_{n+\frac{1}{24}} & x^4_{n+\frac{1}{24}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{24}} & 12x^2_{n+\frac{1}{24}} \\ 0 & 0 & 2 & 6x_{n+\frac{1}{12}} & 12x^2_{n+\frac{1}{12}} \\ 0 & 0 & 2 & 6x_{n+\frac{1}{8}} & 12x^2_{n+\frac{1}{8}} \end{bmatrix}$$

$$
\left.\begin{array}{r}
\left(1+\alpha x_n+\frac{\alpha^2 x_n^2}{2!}+\frac{\alpha^3 x_n^3}{3!}+\frac{\alpha^4 x_n^4}{4!}+\frac{\alpha^5 x_n^5}{5!}\right) \\
\left(1+\alpha x_{n+\frac{1}{24}}+\frac{\alpha^2 x_{n+\frac{1}{24}}^2}{2!}+\frac{\alpha^3 x_{n+\frac{1}{24}}^3}{3!}+\frac{\alpha^4 x_{n+\frac{1}{24}}^4}{4!}+\frac{\alpha^5 x_{n+\frac{1}{24}}^5}{5!}\right) \\
\left(\alpha^2+\alpha^3 x_n+\frac{\alpha^4 x_n^2}{2!}+\frac{\alpha^5 x_n^3}{3!}\right) \\
\left(\alpha^2+\alpha^3 x_{n+\frac{1}{24}}+\frac{\alpha^4 x_{n+\frac{1}{24}}^2}{2!}+\frac{\alpha^5 x_{n+\frac{1}{24}}^3}{3!}\right) \\
\left(\alpha^2+\alpha^3 x_{n+\frac{1}{12}}+\frac{\alpha^4 x_{n+\frac{1}{12}}^2}{2!}+\frac{\alpha^5 x_{n+\frac{1}{12}}^3}{3!}\right) \\
\left(\alpha^2+\alpha^3 x_{n+\frac{1}{8}}+\frac{\alpha^4 x_{n+\frac{1}{8}}^2}{2!}+\frac{\alpha^5 x_{n+\frac{1}{8}}^3}{3!}\right)
\end{array}\right]
$$

By Simplifying some notation in equation (4) and solve for $a_j^{\prime}s,\ j=0,1,2,3,$ 4 and 5 using Gaussians elimination method and substituting the value of $a_j^{\prime}s$ into the equation (2) gives a continuous implicit hybrid block method of the form:

$$
Y(x)=\alpha_0(x)y_n+\alpha_{\frac{1}{24}}(x)y_{n+\frac{1}{24}}+h^2\left[\sum_{j=0}^{\frac{1}{8}}\beta_j(x)f_{n+j}\right], \tag{5}
$$

where $j=0,\frac{1}{24},\frac{1}{12}$, and $\frac{1}{8}$, α_j and β_j represent continuous coefficients expressed as functions of q, where

$$
q=\frac{(x-x_n)}{h},\quad \frac{\mathrm{d}q}{\mathrm{d}t}=\frac{1}{h}. \tag{6}
$$

The coefficients α_j and β_j are given as:

$$
\begin{aligned}
&\alpha_0(q)=1-24q,\\
&\alpha_{\frac{1}{24}}(q)=24q,\\
&\beta_0(q)=-\frac{1}{8640}h^2q\left(995\,328q^4-414\,720q^3+63\,360q^2-4320q+97\right),\\
&\beta_{\frac{1}{24}}(q)=\frac{1}{1440}h^2q\left(497\,664q^4-172\,800q^3+17\,280q^2-19\right),\\
&\beta_{\frac{1}{12}}(q)=-\frac{1}{2880}h^2q\left(995\,328q^4-276\,480q^3+17\,280q^2-13\right),\\
&\beta_{\frac{1}{8}}(q)=\frac{1}{1080}h^2q\left(124\,416q^4-25\,920q^3+1440q^2-1\right).
\end{aligned} \tag{7}
$$

Evaluating equation (7) at $p=\frac{1}{12}$ and $p=\frac{1}{8}$, we get

$$
y_{n+\frac{1}{12}}-2y_{n+\frac{1}{24}}+y_n=\frac{1}{6912}h^2\left(f_n+f_{n+\frac{1}{12}}+10f_{n+\frac{1}{24}}\right), \tag{8}
$$

$$y_{n+\frac{1}{8}} - 3y_{n+\frac{1}{24}} + 2y_n = \frac{1}{6912}h^2\left(2f_n + f_{n+\frac{1}{8}} + 12f_{n+\frac{1}{12}} + 21f_{n+\frac{1}{24}}\right). \quad (9)$$

The first derivative of the equation (7) yields

$$\begin{aligned}
\alpha_0'(q) &= -\frac{24}{h},\\
\alpha_{\frac{1}{24}}'(q) &= \frac{24}{h},\\
\beta_0'(q) &= -\frac{1}{8640}h\left(4976\,640q^4 - 1658\,880q^3 + 190\,080q^2 - 8640q + 97\right),\\
\beta_{\frac{1}{24}}'(q) &= \frac{1}{1440}h\left(2488\,320q^4 - 691\,200q^3 + 51\,840q^2 - 19\right),\\
\beta_{\frac{1}{12}}'(q) &= -\frac{1}{2880}h\left(4976\,640q^4 - 1105\,920q^3 + 51\,840q^2 - 13\right),\\
\beta_{\frac{1}{8}}'(q) &= \frac{1}{1080}h\left(622\,080q^4 - 103\,680q^3 + 4320q^2 - 1\right).
\end{aligned} \quad (10)$$

By evaluating equation (10) at points $p = 0, \frac{1}{24}, \frac{1}{12}$ and $\frac{1}{8}$, we get:

$$hy_n' - 24y_{n+\frac{1}{24}} + 24y_n = \frac{-1}{8640}h^2\left(97f_n + 8f_{n+\frac{1}{8}} - 39f_{n+\frac{1}{12}} + 114f_{n+\frac{1}{24}}\right), \quad (11)$$

$$hy_{n+\frac{1}{24}}' - 24y_{n+\frac{1}{24}} + 24y_n = \frac{1}{8640}h^2\left(38f_n + 7f_{n+\frac{1}{8}} - 36f_{n+\frac{1}{12}} + 171f_{n+\frac{1}{24}}\right), \quad (12)$$

$$hy_{n+\frac{1}{12}}' - 24y_{n+\frac{1}{24}} + 24y_n = \frac{1}{8640}h^2\left(23f_n - 8f_{n+\frac{1}{8}} + 159f_{n+\frac{1}{12}} + 366f_{n+\frac{1}{24}}\right), \quad (13)$$

$$hy_{n+\frac{1}{8}}' - 24y_{n+\frac{1}{24}} + 24y_n = \frac{1}{8640}h^2\left(38f_n + 127f_{n+\frac{1}{8}} + 444f_{n+\frac{1}{12}} + 291f_{n+\frac{1}{24}}\right). \quad (14)$$

3. Modified Block Method

By following Adeniran and Ogundare [17] we define the modified block method as follow:

$$P^0 h^\lambda {Y_m}^{(n)} = h^\lambda \sum_{j=0}^{k} P^{(j)} {Y_{m-j}}^{(n)} + h^\mu \sum_{j=0}^{k} Q^{(j)} F_{m-j}. \tag{15}$$

where n represents power of the derivative, μ is the order of the differential equation. P^0 and P^j are $R \times R$ identity matrices and λ represents the power of h relative to the derivative of the differential equation. Also

$$\begin{aligned} h^\lambda {Y_m}^{(n)} &= \Big[y_{n+\frac{1}{24}}, ..., y_{n+\frac{1}{8}}, ...hy'_{n+\frac{1}{24}}, ..., hy'_{n+\frac{1}{8}}, h^2 y''_{n+\frac{1}{24}}, ..., h^2 y''_{n+\frac{1}{8}} + ..., \\ &\quad h^n y^n_{n+m} \Big]^T, \\ h^\lambda {Y_{m-j}}^{(n)} &= \Big[y_{n-\frac{1}{24}}, ..., y_n, ...hy'_{n-\frac{1}{24}}, ..., hy'_n, h^2 y''_{n-\frac{1}{24}}, ..., h^2 y''_n + ..., h^n y^n_{n+m} \Big]^T, \\ F_{m-j} &= \Big[f_{n-\frac{1}{24}}, f_{n-\frac{1}{12}}, f_{n-\frac{1}{8}} ..., f_m, f_{n+\frac{1}{24}}, f_{n+\frac{1}{12}}, f_{n+\frac{1}{8}} ..., f_m \Big]^T. \end{aligned}$$

By evaluating equations (9), (11), (12), (13), (14) and (15) for $h^\lambda y^\lambda_{n+m}$, $m = 0, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}$ we obtain:

$$P^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P^j = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \frac{h}{24} \\ 0 & 0 & 1 & 0 & 0 & \frac{h}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{h}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

$$Q^j = \begin{bmatrix} 0 & 0 & \frac{97h^2}{207360} & \frac{19h^2}{34560} & -\frac{13h^2}{69120} & \frac{h^2}{25920} \\ 0 & 0 & \frac{7h^2}{6480} & \frac{11h^2}{4320} & -\frac{h^2}{4320} & \frac{h^2}{12960} \\ 0 & 0 & \frac{13h^2}{7680} & \frac{3h^2}{640} & \frac{3h^2}{2560} & \frac{h^2}{3840} \\ 0 & 0 & \frac{h}{64} & \frac{19h}{576} & -\frac{5h}{576} & \frac{h}{576} \\ 0 & 0 & \frac{h}{72} & \frac{h}{18} & \frac{h}{72} & 0 \\ 0 & 0 & \frac{h}{64} & \frac{h}{64} & \frac{h}{64} & \frac{h}{64} \end{bmatrix}.$$

Substituting P^0, P^j and Q^0 of equation (16) into the equation (15) and after much simplification we get:

$$y_{n+\frac{1}{24}} = y_n \quad (17)$$

$$+\frac{1}{24}hy_n^{'} + \frac{1}{207360}h^2\left(97f_n + 8f_{n+\frac{1}{8}} - 39f_{n+\frac{1}{12}} + 114f_{n+\frac{1}{24}}\right), \quad (18)$$

$$y_{n+\frac{1}{12}} = y_n + \frac{1}{12}hy_n^{'} + \frac{1}{12960}h^2\left(14f_n + f_{n+\frac{1}{8}} - 3f_{n+\frac{1}{12}} + 33f_{n+\frac{1}{24}}\right), \quad (19)$$

$$y_{n+\frac{1}{8}} = y_n + \frac{1}{8}hy_n^{'} + \frac{1}{7680}h^2\left(13f_n + 2f_{n+\frac{1}{8}} + 9f_{n+\frac{1}{12}} + 36f_{n+\frac{1}{24}}\right), \quad (20)$$

$$y^{'}_{n+\frac{1}{24}} = y_n^{'} + \frac{1}{576}h\left(9f_n + f_{n+\frac{1}{8}} - 5f_{n+\frac{1}{12}} + 19f_{n+\frac{1}{24}}\right), \quad (21)$$

$$y^{'}_{n+\frac{1}{12}} = y_n^{'} + \frac{1}{72}h\left(f_n + f_{n+\frac{1}{12}} + 4f_{n+\frac{1}{24}}\right), \quad (22)$$

$$y^{'}_{n+\frac{1}{8}} = y_n^{'} + \frac{1}{64}h\left(f_n + f_{n+\frac{1}{8}} + 3f_{n+\frac{1}{12}} + 3f_{n+\frac{1}{24}}\right). \quad (23)$$

4. Analysis of the Proposed Method

Here, we present the analysis of the basic properties of a new proposed method which includes: zero stability, region of absolute stability, order, error constants, consistency and convergence of our new method.

4.1. Zero Stability of the Modified Block Method

To determine the zero stability of the proposed block method, we consider the following condition: According to Adeniran and Ogundare [17], a modified block (15) is said to be zero stable if $R \times R$ be an identity matrix, if $h^{\lambda} \longrightarrow 0$, $\left|\left[\lambda P^{(0)} - P^{(j)}\right]\right| = 0$ and for those roots with $|R_j| \leq 1$, the multiplicity must not exceed the order of the differential equation.

For our new proposed method,

$$\left|\left[\lambda P^{(0)} - P^{(j)}\right]\right| = \left|\begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & -\frac{h}{24} \\ 0 & \lambda & 0 & 0 & 0 & -\frac{h}{12} \\ 0 & 0 & \lambda - 1 & 0 & 0 & -\frac{h}{8} \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix}\right| = 0. \tag{24}$$

$$\lambda^4 \left(1 - \lambda\right)^2 = 0 \tag{25}$$
$$\lambda = 0, 1, 1.$$

If $h \longrightarrow 0$ in equation (15), then equation (18) becomes:

$$\left|\begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{bmatrix}\right| = 0. \tag{26}$$

$$\lambda^2 \left(1 - \lambda\right) = 0 \tag{27}$$
$$\lambda = 0, 1.$$

Since the condition stated above is satisfied, this confirmed that our proposed block method is zero stable.

4.2. Region of Absolute Stability of the Block Method

By following Areo and Adeniyi [18] and Areo and Omojola [23], we formulate the stability matrix as follow:

$$M(z) = V + zB(M - zA)^{-1}U, \tag{28}$$

and the stability function

$$p(\eta, z) = det(\eta I - M(z)). \tag{29}$$

Hence, we represent modified block (15) at $h \longrightarrow 0$ as follow:

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^2 f(y) \\ --- \\ Y_{i-1} \end{bmatrix}, \tag{30}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{97}{207360} & \frac{19}{34560} & -\frac{13}{69120} & \frac{1}{25920} \\ \frac{7}{6480} & \frac{11}{4320} & -\frac{1}{4320} & \frac{1}{12960} \\ \frac{13}{7680} & \frac{3}{640} & \frac{3}{2560} & \frac{1}{3840} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{97}{207360} & \frac{19}{34560} & -\frac{13}{69120} & \frac{1}{25920} \\ \frac{7}{6480} & \frac{19}{34560} & -\frac{1}{4320} & \frac{1}{12960} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{24}} \\ y_{n+\frac{1}{12}} \\ y_{n+\frac{1}{8}} \end{bmatrix}, \quad f(y) = \begin{bmatrix} f_n \\ f_{n+\frac{1}{24}} \\ f_{n+\frac{1}{12}} \\ f_{n+\frac{1}{8}} \end{bmatrix},$$

$$Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{24}} \\ y_n \end{bmatrix}, \quad Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{24}} \\ y_{n+\frac{1}{8}} \end{bmatrix}.$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{31}$$

On putting the value of A, B, U, V, M and I in equations (27) and (28), we get the stability polynomial of the step method which is then coded in MATLAB (R2012a) environment. The needed absolute stability region of our new proposed method is shown in the figure below:

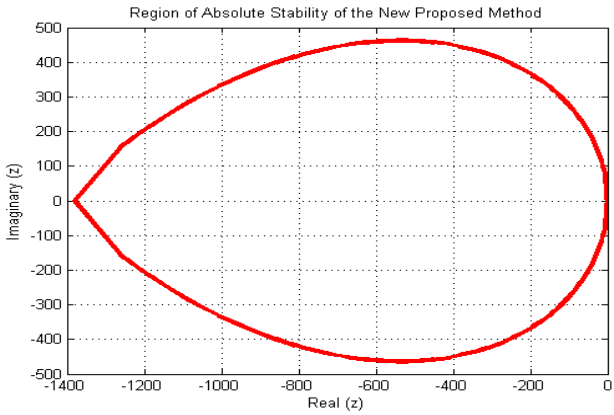

Figure 1: Region of Absolute Stability of the New Proposed Method

From the figure above, it is easy to see that our new method is A-Stable since the plot generated covers so much region of the complex plane $z \in \mathbf{C}^n$.

4.3. Order and Error Constant of the Block Method

Following the method presented by Fatunla [19], we define the linear difference operator as follow:

$$L[y(x);h] = \sum_{j=0}^{k} \left[\alpha_j y\left(x+jh\right) - h^2 \beta_j y''\left(x+jh\right)\right]. \tag{32}$$

If we assume that $y(x)$ has higher derivatives, we can expand the term in equation (32) as a Taylor series about the point x to obtain the following equation:

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_1 h^2 y''(x) ... + C_q h^q y^q(x), \tag{33}$$

where

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j\right], \text{ where } q = 0,1,2,3,...n. \tag{34}$$

As stated by Areo and Omojola [23], the linear operator and the associated block method are said to be of order p if $K_0 = K_1 = ... = K_p = K_{p+1} = 0, K_{p+2} \neq 0$. K_{p+2} is called the error constant. We carry out Taylor series expansion on equations (17), (18), (19), (20), (21) and (22) to get the order of our new proposed block methods as $(4,4,4,4,4,4)$ and error constants as

$$\left(-\frac{7}{91729428480}, -\frac{1}{5733089280}, -\frac{1}{3397386240}, -\frac{19}{5733089280}, -\frac{1}{716636160}, -\frac{1}{212336640}\right).$$

4.4. Consistency and Convergence of the Proposed Method

A method is consistent if the order of the method is greater than one. For this reason, our new proposed method is consistent. According to Jator [8], the two sufficient conditions for a linear hybrid multistep methods to be convergent are to be zero-stable and consistent. Since the two conditions are satisfied. Hence, our new proposed one-eight step hybrid block method for direct solution of second order initial value problems is convergent.

5. Numerical Experiments

Here, we implementation the new developed hybrid block methods to solve initial value problems (IVP) of second order ODEs. The method is coded in MATLAB (R2012a) version environment using window 8.1 as an operating system. The new derived method is tested on some problems to determine the accuracy of the new proposed method and our solutions are compared with the results of other notable researchers in the area of Numerical Analysis. The following initial values problems are consider as numerical examples.

Problem 1: Simple Harmonic Motion

An object stretches a spring 6 inches in equilibrium.

(a) Set up the equation of motion and find its general solution.

(b) Find the displacement of the object for $t > 0$, if it's initially displaced 18 inches above equilibrium and given a downward velocity of $3\frac{ft}{s}$.

From Newton's second law of motion, we have

$$my'' + cy' + ky = F \tag{35}$$

By setting $c = 0$ and $F = 0$, we get

$$my'' + ky = 0 \quad \Rightarrow \quad y'' + \frac{k}{m}y = 0 \tag{36}$$

The equation of the weight of the object is given as follow:

$$mg = k\triangle l \quad \Rightarrow \quad \frac{k}{m} = \frac{g}{\triangle l}, \tag{37}$$

On putting $g = 32\frac{ft}{s^2}$, $\triangle l = \frac{6}{12}ft$ in equation (37) we obtain

$$\frac{k}{m} = \frac{32}{\frac{6}{12}} = 64, \tag{38}$$

Substituting equation (38) into the equation (36) we get

$$y'' + 64y = 0 \tag{39}$$

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus, $y(0) = \frac{3}{2}$, $y'(0) = -3$ and $h = 0.1$. We make use of Maple 17 software to solve equation (39) as follow

$$dsolve\left(\left\{\ddot{y}\,(t) + 64y(t) = 0,\ y\,(0) = \frac{3}{2}, \dot{y}\,(0) = -3\right\}\right). \tag{40}$$

We obtain the exact solution as

$$y\,(t) = -\frac{3}{8}\sin\,(8t) + \frac{3}{2}\cos\,(8t) \tag{41}$$

Problem 2: Critically Damped Motion

Suppose a 64lb weight stretches a spring 6 inches in equilibrium and a dash pot provides a damping force of 32lb for each $\frac{ft}{sec}$ of velocity.

(a) Write the equation of motion of the object for which the motion is critically damped.

(b) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 0$ and $y'(0) = 20$, $h = 0.1$.

Table 1: This table shows the results of test problem 1

t-value	Exact Solution	Numerical Solution	Error in our Method	CPU Time
0.1000	0.77605152993343	0.77605186489576	3.349623e-07	0.0442
0.2000	-0.41863938459250	-0.41863774747164	1.637121e-06	0.0736
0.3000	-1.35938926601855	-1.35938599445251	3.271566e-06	0.0978
0.4000	-1.47555185990679	-1.47554826201357	3.597893e-06	0.1227
0.5000	-0.69666449555495	-0.69666313664636	1.358909e-06	0.1502
0.6000	0.50481020347260	0.50480728922455	2.914248e-06	0.1756
0.7000	1.40007380696749	1.40006708432732	6.722640e-06	0.2002
0.8000	1.44607142631836	1.44606436743660	7.058882e-06	0.2254
0.9000	0.61490152285497	0.61489886852439	2.654331e-06	0.2536
1.0000	-0.58925939319666	-0.58925478764840	4.605548e-06	0.2844

It follows from the equation of motion

$$my'' + cy' + ky = F \tag{42}$$

Substituting $m = 2$, $k = \frac{64}{0.5} = 128\frac{lb}{ft}$, $c = 32\text{lb}$ and $F = 0$ in equation (42) yields

$$y'' + 16y' + 64y = 0 \tag{43}$$

We also use Maple 17 software to get the exact solution of equation (43) as follow

$$dsolve\left(\left\{y''(t) + 16y'(t) + 64y(t) = 0,\ y(0) = 0, y'(0) = 20\right\}\right). \tag{44}$$

The exact solution to the equation (44) is:

$$y(t) = e^{-8t}(1 + 28t)$$

Problem 3: We consider a test problem which was also solved by Awari [21].

$$y'' = -y,\ \text{with initial condition } y'(0) = 1, y(0) = 1, h = 0.1$$

Exact solution:

$$y(x) = \cos x + \sin x.$$

Problem 4: We also consider test problem which was also answered by Awari [21].

$$y'' = 100y,\ \text{with initial condition } y'(1) = -10, y(1) = 1, h = 0.01$$

Exact solution:

$$y(x) = e^{-10x}$$

Problem 5: We also consider a test problem which was also solved by Badmus [22].

$$y'' = y + xe^{3x},\ \text{with initial condition } y'(0) = \frac{-5}{32}, y(0) = \frac{-3}{32}, h = 0.01$$

Exact solution:

$$y(x) = \frac{4x - 3}{32e^{-3x}}.$$

Table 2: This table shows the results of test problem 2

t-value	Exact Solution	Numerical Solution	Error in our Method	CPU Time
3.0000	0.00000000320886	0.00000000313259	7.627261e-11	0.9991
3.1000	0.00000000148933	0.00000000145090	3.843052e-11	1.0304
3.2000	0.00000000069054	0.00000000067126	1.927790e-11	1.0553
3.3000	0.00000000031987	0.00000000031024	9.630866e-12	1.0833
3.4000	0.00000000014804	0.00000000014324	4.793171e-12	1.1105
3.5000	0.00000000006845	0.00000000006608	2.377102e-12	1.1383
3.6000	0.00000000003163	0.00000000003045	1.175015e-12	1.1648
3.7000	0.00000000001460	0.00000000001402	5.790320e-13	1.2014
3.8000	0.00000000000674	0.00000000000645	2.845176e-13	1.2608
3.9000	0.00000000000311	0.00000000000297	1.394245e-13	1.2863
4.0000	0.00000000000143	0.00000000000136	6.814933e-14	1.3132

Table 3: This table shows the results and comparison of test problem 3

X-value	Exact Solution	Numerical Solution	Error in our Method	Error in [21]	CPU Time
0.1000	1.09483758192485	1.09483758191065	1.4204E-11	1.1570E-07	0.0433
0.2000	1.17873590863630	1.17873590858102	5.5286E-11	3.0990E-07	0.0683
0.3000	1.25085669578695	1.25085669566753	1.1942E-10	5.0550E-07	0.0966
0.4000	1.31047933631154	1.31047933610923	2.0230E-10	6.9570E-07	0.1224
0.5000	1.35700810049458	1.35700810019434	3.0023E-10	8.7890E-07	0.1445
0.6000	1.38997808830471	1.38997808789940	4.0532e-10	1.0540E-06	0.1711
0.7000	1.40905987452218	1.40905987400691	5.1527E-10	1.0080E-06	0.2031
0.8000	1.41406280024669	1.41406279962288	6.2381E-10	9.2260E-07	0.3830
0.9000	1.40493687789815	1.40493687717255	7.2560E-10	8.2610E-07	0.4075
1.0000	1.38177329067604	1.38177328986065	8.1538E-10	7.2160E-07	0.4323
1.1000	1.34480348148701	1.34480348059896	8.8806E-10	6.0990E-07	0.5115
1.2000	1.29439684044390	1.29439683950511	9.3879E-10	4.9190E-07	0.5367

Table 4: This table shows the results and comparison of test problem 4

X-value	Exact Solution	Numerical Solution	Error in our Method	Error in [21]	CPU Time
0.0100	0.90483741803596	0.90483741805021	1.4251E-11	1.3530E-07	0.0443
0.0200	0.81873075307798	0.81873075313401	5.6031E-11	3.6580E-07	0.0721
0.0300	0.74081822068172	0.74081822080487	1.2315E-10	6.0510E-07	0.0975
0.0400	0.67032004603564	0.67032004624957	2.1393E-10	8.5020E-07	0.1224
0.0500	0.60653065971263	0.60653066003977	3.2713E-10	1.1040E-06	0.1431
0.0600	0.54881163609403	0.54881163655600	4.6197E-10	1.3690E-06	0.1704
0.0700	0.49658530379141	0.49658530440945	6.1804E-10	1.4500E-06	0.1974
0.0800	0.44932896411722	0.44932896491362	7.9640E-10	1.5970E-06	0.2213
0.0900	0.40656965974060	0.40656966073477	9.9417E-10	1.7630E-06	0.2470
0.1000	0.36787944117144	0.36787944238672	1.2153E-09	1.9460E-06	0.2720
0.1100	0.33287108369808	0.33287108515775	1.4597E-09	2.0990E-06	0.2987
0.1200	0.30119421191220	0.30119421364096	1.7288E-09	2.3740E-06	0.3242

Table 5: This table shows the results and comparison of test problem 5

X-value	Exact Solution	Numerical Solution	Error in our Method	Error in [22],$k = 4$, Case II	CPU Time
1.003125	1.00307652585770	1.00307652586311	5.411893E-12	8.300000E-08	0.1102
1.006250	1.00605750308352	1.00605750310550	2.197598E-11	1.160000E-06	0.1673
1.009375	1.00894499508884	1.00894499513798	4.913780E-11	6.638000E-06	0.2002
1.012500	1.01174101816800	1.01174101825442	8.641976E-11	9.491000E-06	0.2349
1.015625	1.01444754268642	1.01444754281979	1.333629E-10	1.953500E-06	0.2681
1.018750	1.01706649423568	1.01706649442521	1.895248E-10	9.416000E-06	0.2933
1.021875	1.01959975475630	1.01959975501078	2.544809E-10	4.650500E-05	0.3281
1.025000	1.02204916362945	1.02204916395727	3.278211E-10	4.712200E-05	0.3662
1.028125	1.02441651873842	1.02441651914757	4.091472E-10	1.869260E-04	0.4053
1.031250	1.02670357750082	1.02670357799891	4.980834E-10	4.433210E-04	0.4324

Problem 6: We consider a test problem which also answered by Kayode and Adeyeye [10].

$$y'' = \left(\frac{y'^2}{2y} - 2y\right), \text{ with initial condition } y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, y\left(\frac{\pi}{6}\right) = \frac{1}{4}, h = 0.01$$

Exact solution:

$$y(x) = \sin^2 x.$$

6. Explanation of the Results

In this article, we have applied the techniques of collocation and interpolation to derive a uniform fourth order continuous one-eight step hybrid block method for solving second order initial value problems of ODEs directly. We have solved two real-life problems to determine the performance of the new proposed method and the numerical solutions were shown in the tables 1 and 2. The result from table 3, table 4 and table 5 displayed that our new method is more efficient than the methods proposed by Awari [21] and Badmus [22]. It can also be seen from the table 6 that new proposed method gives better solutions than the solutions presented by Kayode and Adeyeye, despite the high order of their methods, new block method of order four is more accurate than their predictor-corrector method of order six.

7. Conclusion

We have introduced a new step continuous hybrid block method that give solutions to second order initial value problems of ODEs directly without the rigor of reducing to an equivalent system of first order. We used our new developed method to solve six numerical examples which are in the form of equation (1) and the numerical solutions acquired were much better when compared with the results of other researchers in the tables 3, 4, 5 and 6. We conclude that our new proposed continuous hybrid block method gives better approximate solutions and more efficient than some existing methods.

Table 6: This table shows the results and comparison of test problem 6

X-value	Exact Solution	Numerical Solution	Error in our Method	Error in [10]	CPU Time
0.5440	0.26788483005162	0.26788483012353	7.1911E-11	4.0400E-10	0.2327
0.5540	0.27678780616404	0.27678780632585	1.6180E-10	1.1000E-09	0.2855
0.5640	0.28578006417787	0.28578006446642	2.8854E-10	2.0200E-09	0.3013
0.5740	0.29485800730980	0.29485800776253	4.5273E-10	3.1700E-09	0.3170
0.5840	0.30401800450362	0.30401800515852	6.5490E-10	4.5500E-09	0.3323
0.5940	0.31325639188258	0.31325639277813	8.9555E-10	6.1500E-09	0.3685
0.6040	0.32256947421490	0.32256947539001	1.1751E-09	7.9700E-09	0.3861
0.6140	0.33195352639183	0.33195352788574	1.4939E-09	9.9900E-09	0.4158
0.6240	0.34140479491761	0.34140479677005	1.8524E-09	1.2200E-08	0.4468

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