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HOPF BIFURCATION IN A DELAYED RAMSEY MODEL WITH VON BERTALANFFY POPULATION LAW

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Abstract: We analyze a modified version of the Ramsey model with a lag between investment and production. The associated characteristic equation contains an additional exponential factor that opening up the possibility of complex roots. The existence of a Hopf bifurcation near the positive equilibrium is demonstrated.

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Key Words: Ramsey model, von Bertalanffy population, Hopf bifurcation

1. Introduction

Recently Accinelli and Brida [1] have studied the dynamics of the Ramsey model with the law describing population growth given by the von Bertalanffy population growth law. They find that the equilibrium is a saddle point with two dimensional basins of attraction. In this paper, we propose to investigate their model when capital takes time to became productive (in the literature referred as time to build). This implies having capital described by a delay differential equation. It is proved that time to build is responsible of the oscillatory behavior of the main variables and that exists parametrization under which the dynamics of the economy can be a cycle. In contrast to standard Ramsey model, a Hopf bifurcation is generated at the positive equilibrium as

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the delay increases. For future research we aim to introduce the time to build assumption to other economic growth models (e.g., Ferrara and Guerrini [2-4], Guerrini [5-12]).

2. The Model

The model and the notation are the same as Accinelli and Brida [1], where the Ramsey growth model is reformulated using the von Bertalanffy equation [15] to model population growth. Within this framework, an application of the Pontryagin Maximum Principle [14] yields that the dynamic evolution of the economy is described by

$$\dot{k}_t = k_{t-T}^{\alpha} - \left[r\left(\frac{L_\infty}{L_t} - 1\right) + \delta \right] k_t - c_t, \tag{1}$$

$$\dot{c}_t = \sigma c_t \left\{ \alpha k_t^{\alpha-1} - \left[r\left(\frac{L_\infty}{L_t} - 1\right) + \delta + \rho \right] \right\}, \tag{2}$$

$$\dot{L}_t = r(L_\infty - L_t), \tag{3}$$

where c_t denotes consumption, k_t is capital, L_t indicates population, L_∞ is the theoretical maximum asymptote size of population and r is the constant determining the speed at which population reaches the asymptote. Moreover, $\rho > 0$ is the time preference rate, σ represents the constant intertemporal elasticity of substitution, $\alpha \in (0,1)$ is capital's share and $\delta > 0$ denotes the rate of capital depreciation. Unlike Accinelli and Brida [1], we now modify the model by allowing for a time delay $T > 0$ in the productive operation of installed capital as well as in capital depreciation. As a result, Eqs. $(1-3)$ are replaced by the following system of differential equations:

$$\dot{k}_t = k_{t-T}^{\alpha} - \left[r\left(\frac{L_\infty}{L_t} - 1\right) + \delta \right] k_{t-T} - c_t, \tag{4}$$

$$\dot{c}_t = \sigma c_t \left\{ \alpha k_{t-T}^{\alpha-1} - \left[r\left(\frac{L_\infty}{L_t} - 1\right) + \delta + \rho \right] \right\}, \tag{5}$$

$$\dot{L}_t = r(L_\infty - L_t), \tag{6}$$

with $k_t = \varphi_t$, $t \in [-T, 0]$ being the initial capital function. It is clear that the critical points of Eqs. $(4-6)$ correspond to those with vanishing delay. Hence, there exists a unique non-trivial steady state:

$$k_* = \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}, \quad c_* = k_*^{\alpha} - \delta k_*, \quad L_* = L_\infty.$$

3. Existence of Hopf Bifurcation

The corresponding linear system of system $(4-6)$ at (k_*, c_*, L_*) is in the form

$$\dot{k}_t = -(c_t - c_*) + \frac{rk_*}{L_*}(L_t - L_*) + \rho(k_{t-T} - k_*), \tag{7}$$

$$\dot{c}_t = \frac{\sigma c_* r}{L_*}(L_t - L_*) + (\alpha - 1)\alpha\sigma c_* k_*^{\alpha-2}(k_{t-T} - k_*), \tag{8}$$

$$\dot{L}_t = -r(L_t - L_*). \tag{9}$$

Then the characteristic equation of $(7-9)$ is

$$\det \begin{bmatrix} -\lambda + \rho e^{-\lambda T} & -1 & \frac{rk_*}{L_*} \\ (\alpha-1)\alpha\sigma c_* k_*^{\alpha-2} e^{-\lambda T} & -\lambda & \frac{\sigma c_* r}{L_*} \\ 0 & 0 & -r-\lambda \end{bmatrix} = 0,$$

which gives

$$(-r-\lambda)\left[\lambda^2 - \rho\lambda e^{-\lambda T} - (1-\alpha)\alpha\sigma c_* k_*^{\alpha-2} e^{-\lambda T}\right] = 0. \tag{10}$$

When $T = 0$, equation (7) has two roots with negative real parts and a root with positive real part (Accinelli and Brida [1]). Next, we shall explore the distribution of characteristic roots when $T > 0$. Let $\lambda = i\omega$ $(\omega > 0)$ be a root of equation (10). Then

$$\omega^2 + \rho i\omega e^{-i\omega T} + (1-\alpha)\alpha\sigma c_* k_*^{\alpha-2} e^{-i\omega T} = 0.$$

Separation of the real and imaginary parts yields

$$\rho\omega \sin\omega T + (1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\cos\omega T = -\omega^2, \tag{11}$$

$$\rho\omega\cos\omega T - (1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\sin\omega T = 0. \tag{12}$$

Hence, we obtain

$$\omega^4 - \rho^2\omega^2 - \left[(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]^2 = 0. \tag{13}$$

Lemma 1. *Equation* (10) *has a pair of purely imaginary roots* $\pm i\omega_0$ *when* $T = T_0$, *where*

$$\omega_0 = \sqrt{\frac{\rho^2 + \sqrt{\rho^4 + 4\left[(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]^2}}{2}},$$

$$T_0 = \frac{1}{\omega_0} arc\cos\left\{ -\frac{\omega_0^2(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}}{\rho^2\omega_0^2 + \left[(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]^2} \right\}.$$

Proof. It is easy to verify that equation (13) has the unique positive root ω_0. This means that the characteristic equation only has a pair of purely imaginary roots. Moreover, eliminating $\sin\omega T$ from (11), (12) gives

$$\cos\omega_0 T = -\frac{\omega_0^2(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}}{\rho^2\omega_0^2 + \left[(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]^2}.$$

This completes the proof. □

Lemma 2. *Let $\lambda(T) = \mu(T) + i\omega(T)$ be the root of (10) satisfying $\mu(T_0) = 0$ and $\omega(T_0) = \omega_0$. Then*

$$\left.\frac{d\,(Re\lambda)}{dT}\right|_{\lambda=i\omega_0} > 0.$$

Proof. By substituting $\lambda(T)$ into equation (10) and differentiating both sides of the equation with respect to T, we derive

$$\left(\frac{d\lambda}{dT}\right)^{-1} = -\frac{2e^{\lambda T}}{\rho\lambda + (1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}} + \frac{\rho}{\lambda\left[\rho\lambda + (1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]} - \frac{T}{\lambda}.$$

Consequently, we get

$$Re\left(\frac{d\lambda}{dT}\right)^{-1}_{T=T_0} = \frac{2\omega_0^2 - \rho^2}{\rho^2\omega_0^2 + \left[(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]^2} > 0.$$

From equation (12) it follows $\omega_0^4 - \rho^2\omega_0^2 = \left[(1-\alpha)\alpha\sigma c_* k_*^{\alpha-2}\right]^2 > 0$, i.e. $\omega_0^2 - \rho^2 > 0$. The statement is now immediate. □

From the previous two Lemmas and the Hopf bifurcation Theorem [13], we have the following result.

Theorem 1. *System $(4-6)$ undergoes a Hopf bifurcation at (k_*, c_*, L_*) when $T = T_0$.*

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