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AGE-STRUCTURED MALARIA TRANSMISSION DYNAMIC MODEL WITH TEMPORAL IMMUNITY IN THE HUMAN POPULATION

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Abstract: We develop and analyse an age-structured model of malaria transmission dynamics taking into account an induced immunity in the human population . After computing the threshold number R_0, we show the local and global asymptotic stability of the disease-free equilibrium state when $R_0 < 1$. The existence of a unique endemic steady state and its local asymptotic stability are showed when $R_0 > 1$.

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Key Words: age-structured model, differential equations, basic reproductive number, asymptotic stability, characteristic lines method

1. Introduction

Malaria, a vector-borne disease caused by the bites of infected female mosquitoes, is responsible of about 429000 deaths annually world wide(2015) according to the World Health Organisation[6]. Its devastating effects are mostly affect-

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ing sub-Saharan region where 80% of deaths occur. Among them, children under five years and pregnant women constitute the most vulnerable groups. Although efforts at governmental level in different African countries are increasing each year, malaria still one of the major public health problem faced by those countries.

In the quest of understanding this strange situation about malaria, mathematical models are providing useful information about transmission dynamics of the disease. Since the works of Ross and MacDonald [5], different extensions of their models have been added. The present model extends the ideas in[1] and the mathematical analysis follows the approaches in [2][3] [7][11] and references therein.

The remainder of the work is given as follows: in Section 2, after defining the parameters involved in the model, we establish our model equations. Section 3 deals with the analysis of the model by computing the basic reproduction number after the determination of the disease free equilibrium state. In Section 4, existence and local stability of the endemic equilibrium state are examined. We conclude our work in Section 5.

2. Model Formulation

The present malaria transmission model takes into consideration the temporal immunity induced in the human population. Total populations are denoted by $N_h(t,a)$ and $N_v(t,a)$ for humans and mosquitoes, respectively. We denote by $S_h(t,a), L_h(t,a), I_h(t,a),\ R_h(t,a)$, respectively for Susceptible, Latent, Infected and Recovered classes in the human population; similarly, we denote by $S_v(t,a),\ L_v(t,a), I_v(t,a)$ for mosquitoes population. Additional susceptible individuals are recruited at birth rate denoted by $\alpha_h(a)$ for humans and $\alpha_v(a)$ for mosquitoes. Susceptible individuals enter the latent classes with forces of infection given by $\varepsilon_h(a)K(t)$ and $\varepsilon_v(a)K(t)$ where $\varepsilon_h(a)$ and $\varepsilon_v(a)$ represent the probability of getting the infection for humans and mosquitoes, respectively. $K(t)$ is the total number of sources of infections. Here, $\beta(a)$ is the biting rate. The rates of progression from latent states to infected states are given by $\delta_h(a)$ and $\delta_v(a)$ for humans and mosquitoes, respectively. A fraction $\lambda(a)$ of infected humans progress to the recovered state. Recovered human individuals return in the susceptible class at a rate $\gamma(a)$. Natural deaths rate are imposed to all classes and are denoted by $\mu_h(a)$ and $\mu_v(a)$ for humans and mosquitoes, respectively. The additional mortality rate induced by malaria is denoted by $\eta(a)$. We take ω as the maximum age attainable between the two populations. We

pictured the general dynamics of malaria transmission by the following diagram:

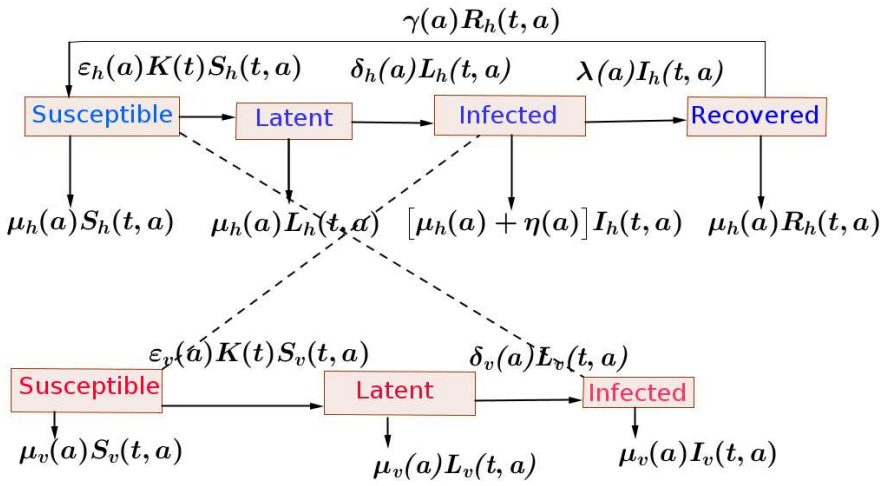

Figure 1: Flow diagram of malaria transmission dynamic

The dynamic is governed by the following system of partial differential equations:

$$\begin{cases} (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})S_h(t,a)=\alpha_h(a)N_h(t,a)+\gamma(a)R_h(t,a) \\ -(\mu_h(a)+\varepsilon_h(a)K(t,a))S_h(t,a), \\ (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})L_h(t,a)=\varepsilon_h(a)K_h(t,a)S_h(t,a)-(\mu_h(a)+\delta_h(a))L_h(t,a), \\ (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})I_h(t,a)=\delta_h(a)L_h(t,a)-(\mu_h(a)+\lambda(a)+\eta(a))I_h(t,a), \\ (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})R_h(t,a)=\lambda(a)I_h(t,a)-(\mu_h(a)+\gamma(a))R_h(t,a), \\ (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})S_v(t,a)=\alpha_v(a)N_v(t,a)-(\mu_v(a)+\varepsilon_v(a)K(t,a))S_v(t,a), \\ (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})L_v(t,a)=\varepsilon_v(a)K(t,a)S_v(t,a)-(\mu_v(a)+\delta_v(a))L_v(t,a), \\ (\frac{\partial}{\partial t}+\frac{\partial}{\partial a})I_v(t,a)=\delta_v(a)L_v(t,a)-\mu_v(a)I_v(t,a), \\ K(t)=\int_0^\omega \beta(a)\frac{I_h(t,a)+\mu I_v(t,a)}{N_h(t,a)}da, \end{cases} \tag{1}$$

where $\mu = N_h(t,a)/N_v(t,a)$ with initial and boundary conditions:

$$\begin{cases} S_h(t,0)=N_h(t,0); L_h(t,0)=I_h(t,0)=R_h(t,0)=0; \\ S_v(t,0)=N_v(t,0); L_v(t,0)=I_v(t,0)=0. \\ S_h(0,a)=S_{0h}(a); L_h(0,a)=L_{0h}(a); I_h(0,a)=I_{0h}(a); R_h(0,a)=R_{0h}(a); \\ S_v(0,a)=S_{0v}(a); L_v(0,a)=L_{0v}(a); I_v(0,a)=I_{0v}(a). \end{cases}$$

We consider the following fractions:

$$\begin{cases} s_h(t,a) = \frac{S_h(t,a)}{N_h(t,a)}, & l_h(t,a) = \frac{L_h(t,a)}{N_h(t,a)}, & i_h(t,a) = \frac{I_h(t,a)}{N_h(t,a)}, \\ r_h(t,a) = \frac{R_h(t,a)}{N_h(t,a)}, & s_v(t,a) = \frac{S_v(t,a)}{N_v(t,a)}, & l_v(t,a) = \frac{L_v(t,a)}{N_v(t,a)}, \\ i_v(t,a) = \frac{I_v(t,a)}{N_v(t,a)}. \end{cases}$$

Introducing the fractions in system (1), we get:

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})s_h(t,a) = & \alpha_h(a) + \gamma(a)r_h(t,a) - (\alpha_h(a) + \varepsilon_h(a)k(t) \\ & -\eta(a)i_h(t,a))s_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})l_h(t,a) = & \varepsilon_h(a)k(t)s_h(t,a) - (\alpha_h(a) + \delta(a) \\ & -\eta(a)i_h(t,a))l_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})i_h(t,a) = & \delta(a)l_h(t,a) - (\alpha_h(a) + \lambda(a) + \eta(a) \\ & -\eta(a)i_h(t,a))i_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})r_h(t,a) = & \lambda(a)i_h(t,a) - (\alpha_h(a) + \gamma(a) - \eta(a)i_h(t,a))r_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})s_v(t,a) = & \alpha_v(a) - (\alpha_v(a) + \varepsilon_v(a)k(t))s_v(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})l_v(t,a) = & \varepsilon_v(a)k(t)s_v(t,a) - (\alpha_v(a) + \delta_v(a))l_v(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})i_v(t,a) = & \delta_v(a)l_v(t,a) - \alpha_v(a)i_v(t,a), \\ k(t) = & \int_0^\omega \beta(a)(i_h(t,a) + i_v(t,a))da, \end{cases} \tag{2}$$

with initial and boundary values:

$$\begin{cases} s_h(t,0) = 1; \quad l_h(t,0) = i_h(t,0) = r_h(t,0) = 0; \\ s_v(t,0) = 1, \quad l_v(t,0) = i_v(t,0) = 0. \\ s_h(0,a) = s_{0h}; \quad l_h(0,a) = l_{0h}; \quad i_h(0,a) = i_{0h}; \\ r_h(0,a) = r_{0h}; \quad s_v(0,a) = s_{0v}; \\ l_v(0,a) = l_{0v}; \quad i_v(0,a) = i_{0v}. \end{cases}$$

From system (2), it is well observed that the population dynamics is affected by the disease. Following similar ideas as in [2][3][7][11] and references therein, it is easy to show that the present model is well posed.

3. Disease Free Equilibrium and Basic Reproductive Number

The disease free equilibrium is obtained from the following time independent system obtained from (2):

$$\begin{cases} \frac{ds'_h(a)}{da} = \alpha_h(a) + \gamma(a)r'_h(a) - (\alpha_h(a) + \varepsilon_h(a)k' - \eta(a)i'_h(a))s'_h(a), \\ \frac{dl'_h(a)}{da} = \varepsilon_h(a)k's'_h(a) - (\alpha_h(a) + \delta_h(a) - \eta(a)i'_h(a))l'_h(a), \\ di'_h(a)da = \delta_h(a)l'_h(a) - (\alpha_h(a) + \lambda(a) + \eta(a) - \eta(a)i'_h(a))i'_h(a), \\ \frac{dr'_h(a)}{da} = \lambda(a)i'_h(a) - (\alpha_h(a) + \gamma(a) - \eta(a)i'h(a))r'_h(a), \\ \frac{ds'_v(a)}{da} = \alpha_v(a) - (\alpha_v(a) + \varepsilon_v(a)k'(a))s'_v(a), \\ \frac{dl'_v(a)}{da} = \varepsilon_v(a)k'(a)s'_v(a) - (\alpha_v(a) + \delta_v(a))l'_v(a), \\ \frac{di'_v(a)}{da} = \delta_v(a)l'_v(a) - \alpha_v(a)i'_v(a), \\ k' = \int_0^\omega \beta(a)(i'_h(a) + i'_v(a))da, \end{cases} \tag{3}$$

and by setting equal to zero all classes containing infections, it follows that the disease free equilibrium is given by $(s'_h, l'_h, i'_h, r'_h, s'_v, l'_v, i'_v) = (1, 0, 0, 0, 1, 0, 0)$.

In order to determine the basic reproductive number, we introduce a perturbation at the disease free equilibrium in (2) as follows:

$$\begin{cases} s_h(t,a) = \bar{s}_h(t,a) + s'_h; l_h(t,a) = \bar{l}_h(t,a) + l'_h; i_h(t,a) = \bar{i}_h(t,a) + i'_h; \\ r_h(t,a) = \bar{r}_h(t,a) + r'_h; s_v(t,a) = \bar{s}_v(t,a) + s'_v; \\ l_v(t,a) = \bar{l}_v(t,a) + l'_v; i_v(t,a) = \bar{i}_v(t,a) + i'_v. \end{cases}$$

The perturbed system is described by the following equations:

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{s}_h(t,a) = \gamma(a)\bar{r}_h(t,a) + \eta(a)\bar{i}_h(t,a) - \alpha_h(a)\bar{s}_h(t,a) - \varepsilon_h(a)\bar{k}(t), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{l}_h(t,a) = \varepsilon_h(a)\bar{k}(t) - (\alpha_h(a) + \delta(a))\bar{l}_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{i}_h(t,a) = \delta(a)\bar{l}_h(t,a) - (\alpha_h(a) + \lambda(a) + \eta(a))\bar{i}_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{r}_h(t,a) = \lambda(a)\bar{i}_h(t,a) - (\alpha_h(a) + \gamma(a))\bar{r}_h(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{s}_v(t,a) = -\alpha_v(a)\bar{s}_v(t,a) - \varepsilon_v(a)\bar{k}(t), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{l}_v(t,a) = \varepsilon_v(a)\bar{k}(t) - (\alpha_v(a) + \delta_v(a))\bar{l}_v(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\bar{i}_v(t,a) = \delta_v(a)\bar{l}_v(t,a) - \alpha_v(a)\bar{i}_v(t,a), \\ \bar{k}(t) = \int_0^\omega (\bar{i}_h(t,a) + \bar{i}_v(t,a))da. \end{cases} \tag{4}$$

with $\bar{s}_h(0,t) = \bar{l}_h(0,t) = \bar{i}_h(0,t) = \bar{r}_h(0,t) = \bar{s}_v(0,t) = \bar{l}_v(0,t) = \bar{i}_v(0,t) = 0$.

We consider solutions in the exponential form as:

$$\begin{cases} \bar{s}_h(t,a) = \bar{s}_h(a)e^{\pi t}; \bar{l}_h(t,a) = \bar{l}_h(a)e^{\pi t}; \bar{i}_h(t,a) = \bar{i}_h(a)e^{\pi t}; \bar{r}_h(t,a) = \bar{r}_h(a)e^{\pi t}; \\ \bar{s}_v(t,a) = \bar{s}_v(a)e^{\pi t}; \bar{l}_v(t,a) = \bar{l}_v(a)e^{\pi t}; \bar{i}_v(t,a) = \bar{i}_v(a)e^{\pi t}. \end{cases}$$

We obtain the linear system given by:

$$\begin{cases} \frac{d\bar{s}_h(a)}{da} = \gamma(a)\bar{r}_h(a) + \eta(a)\bar{i}_h(a) - (\alpha_h(a)+\pi)\bar{s}_h(a) - \varepsilon_h(a)\bar{k}, \\ \frac{d\bar{l}_h(a)}{da} = \varepsilon_h(a)\bar{k} - (\alpha_h(a)+\delta_h(a)+\pi)\bar{l}_h(a), \\ \frac{d\bar{i}_h(a)}{da} = \delta_h(a)\bar{l}_h(a) - (\alpha_h(a)+\lambda(a)+\eta(a)+\pi)\bar{i}_h(a), \\ \frac{d\bar{r}_h(a)}{da} = \lambda(a)\bar{i}_h(a) - (\alpha_h(a)+\gamma(a)+\pi)\bar{r}_h(a), \\ \frac{d\bar{s}_v(a)}{da} = -\varepsilon_v(a)\bar{k}(a) - (\alpha_v(a)+\pi)\bar{s}_v(a), \\ \frac{d\bar{l}_v(a)}{da} = \varepsilon_v(a)\bar{k}(a) - (\alpha_v(a)+\delta_v(a)+\pi)\bar{l}_v(a), \\ \frac{d\bar{i}_v(a)}{da} = \delta_v(a)\bar{l}_v(a) - (\alpha_v(a)+\pi)\bar{i}_v(a), \\ \bar{k} = \int_0^\omega \beta(a)(\bar{i}_h(a)+\bar{i}_v(a))da. \end{cases} \tag{5}$$

We obtain the following expressions for infectious and latent classes:

$$\bar{l}_h(a) = \int_0^a \varepsilon_h(\zeta)\bar{k}e^{-\int_\zeta^a(\alpha_h(\sigma)+\delta_h(\sigma)+\pi)d\sigma}d\zeta, \tag{6}$$

$$\bar{i}_h(a) = \int_0^a \delta_h(\zeta)\bar{l}_h(\zeta)e^{-\int_\zeta^a(\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)+\pi)d\sigma}d\zeta \tag{7}$$

$$\bar{l}_v(a) = \int_0^a \varepsilon_v(\zeta)\bar{k}e^{-\int_\zeta^a(\alpha_v(\sigma)+\delta_v(\sigma)+\pi)d\sigma}d\zeta, \tag{8}$$

$$\bar{i}_v(a) = \int_0^a \delta_v(\zeta)\bar{l}_v(\zeta)e^{-\int_\zeta^a(\alpha_v(\sigma)+\pi)d\sigma}d\zeta. \tag{9}$$

Replacing the expressions of $\bar{l}_h(a)$ in (7) and $\bar{l}_v(a)$ in (9), the infected classes become:

$$\bar{i}_h(a) = \int_0^a \delta_h(\zeta)\int_0^\zeta \varepsilon_h(\theta)\bar{k}e^{-\int_\theta^\zeta(\alpha_h(\sigma)+\delta_h(\sigma)+\pi)d\sigma}d\theta e^{-\int_\zeta^a(\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)+\pi)d\sigma}d\zeta, \tag{10}$$

$$\bar{i}_v(a) = \int_0^a \delta_v(\zeta) \int_0^\zeta \varepsilon_v(\theta)\bar{k}e^{-\int_\zeta^a(\alpha_v(\sigma)+\delta_v(\sigma)+\pi)d\sigma} d\theta e^{-\int_\zeta^a(\alpha_v(\sigma)+\pi)d\sigma} d\zeta. \quad (11)$$

We insert equations (10) and (11) into the last equation of system (5), we get the following characteristic equation

$$1 = \int_0^\omega \beta(a)\Big\{ \int_0^a \delta_h(\zeta) \int_0^\zeta \varepsilon_h(\theta)e^{-\int_\theta^\zeta(\alpha_h(\sigma)+\delta_h(\sigma)+\pi)d\sigma} d\theta e^{-\int_\zeta^a(\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)+\pi)d\sigma} d\zeta + \int_0^a \delta_v(\zeta) \int_0^\zeta \varepsilon_v(\theta)e^{-\int_\zeta^a(\alpha_v(\sigma)+\delta_v(\sigma)+\pi)d\sigma} d\theta e^{-\int_\zeta^a(\alpha_v(\sigma)+\pi)d\sigma} d\zeta \Big\} da. \quad (12)$$

We denote the right hand side of equation (12) by $F(\pi)$ and define our basic reproductive number by $R_0 = F(0)$, which becomes:

$$R_0 = \int_0^\omega \beta(a)\Big\{ \int_0^a \delta_h(\zeta) \int_0^\zeta \varepsilon_h(\theta)e^{-\int_\theta^\zeta(\alpha_h(\sigma)+\delta_h(\sigma))d\sigma} d\theta e^{-\int_\zeta^a(\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma))d\sigma} d\zeta + \int_0^a \delta_v(\zeta) \int_0^\zeta \varepsilon_v(\theta)e^{-\int_\zeta^a(\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\theta e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta \Big\} da. \quad (13)$$

Theorem 1. *The disease free equilibrium is locally asymptotically stable if* $R_0 < 1$.

We note that $F'(\pi) < 0$, and that $\lim_{\pi \longrightarrow -\infty} F(\pi) = +\infty$ and $\lim_{\pi \longrightarrow +\infty} F(\pi) = 0$. The characteristic equation (12) admits a unique real solution which is negative. Using similar arguments as in [7] [11], the local asymptotic stability follows. In what follows, we assume that $e^{\int_0^\omega \eta(\sigma) i_h(t-a+\sigma,\sigma)d\sigma} \leq C$, with $C \geq e$ and $\forall t$.

Theorem 2. *The disease free equilibrium is globally asymptotically stable if* $C * R_0 < 1$.

Proof. In what follows, we show that $\lim_{t \longrightarrow +\infty} s_h(t,a) = s'_h$, $\lim_{t \longrightarrow +\infty} l_h(t,a) = l'_h$, $\lim_{t \longrightarrow +\infty} i_h(t,a) = i'_h$, $\lim_{t \longrightarrow +\infty} r_h(t,a) = r'_h$, $\lim_{t \longrightarrow +\infty} s_v(t,a) = s'_v$, $\lim_{t \longrightarrow +\infty} l_v(t,a) = l'_v$, $\lim_{t \longrightarrow +\infty} i_v(t,a) = i'_v$. Using the characteristics lines method when $t - a > 0$, we get:

$$s_h(t,a) = \int_0^a (\alpha_h(\zeta) + \gamma(\zeta) r_h(t-a+\zeta,\zeta)) \quad (14)$$

$$\times e^{-\int_\zeta^a (\alpha_h(\sigma)+\varepsilon_h(\sigma)k(t-a+\sigma)-\eta(\sigma)i_h(t-a+\sigma,\sigma))d\sigma} d\zeta,$$

$$l_h(t,a) = \int_0^a \varepsilon_h(\zeta) s_h(t-a+\zeta,\zeta) e^{-\int_\zeta^a (\alpha_h(\sigma)+\delta_h(\sigma)-\eta(\sigma)i_h(t-a+\sigma,\sigma))d\sigma} d\zeta, \quad (15)$$

$$i_h(t,a) = \int_0^a \delta_h(\zeta) l_h(t-a+\zeta,\zeta) e^{-\int_\zeta^a (\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma)i_h(t-a+\sigma,\sigma))d\sigma} d\zeta, \quad (16)$$

$$r_h(t,a) = \int_0^a \lambda(\zeta) i_h(t-a+\zeta,\zeta) e^{-\int_\zeta^a (\alpha_h(\sigma)+\gamma(\sigma)-\eta(\sigma)i_h(t-a+\sigma,\sigma))d\sigma} d\zeta, \quad (17)$$

$$s_v(t,a) = \int_0^a \alpha_v(\zeta) e^{-\int_\zeta^a (\alpha_v(\sigma)+\varepsilon_v(\sigma)k(t-a+\sigma))d\sigma} d\zeta, \quad (18)$$

$$l_v(t,a) = \int_0^a \varepsilon_v(\zeta) k(t-a+\zeta) s_v(t-a+\zeta,\zeta) e^{-\int_\zeta^a (\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\zeta, \quad (19)$$

$$i_v(t,a) = \int_0^a \delta_v(\zeta) l_v(t-a+\zeta,\zeta) e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta. \quad (20)$$

We replace the expressions of $l_h(t,a)$ and $l_v(t,a)$ in equations (16) and (19), respectively. We obtain the following expressions of $i_h(t,a)$ and $i_v(t,a)$:

$$i_h(t,a) = \int_0^a \delta_h(\zeta) \int_0^\zeta \varepsilon_h(\theta) k(t-\zeta+\theta) s_h(t-\zeta+\theta,\zeta) \quad (21)$$
$$\times e^{-\int_\theta^\zeta (\alpha_h(\sigma)+\delta_h(\sigma)-\eta(\sigma)i_h(t-\zeta+\sigma,\sigma))d\sigma} d\theta$$
$$\times e^{-\int_\zeta^a (\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma)i_h(t-a+\sigma,\sigma))d\sigma} d\zeta,$$

$$i_v(t,a) = \int_0^a \delta_v(\zeta) \int_0^\zeta \varepsilon_v(\theta) k(t-\zeta+\theta) s_v(t-\zeta+\theta,\zeta) \quad (22)$$
$$\times e^{-\int_\theta^\zeta (\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\theta e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta$$

Using equations (21) and (22) in the expression of $k(t)$, we get:

$$k(t) = \int_0^{\omega} \beta(a)\Big\{ \int_0^a \delta_h(\zeta) \int_0^{\zeta} \varepsilon_h(\theta) k(t-\zeta+\theta) s_h(t-\zeta+\theta,\zeta) \times e^{-\int_\theta^\zeta (\alpha_h(\sigma)+\delta_h(\sigma)-\eta(\sigma) i_h(t-\zeta+\sigma,\sigma))d\sigma} d\theta \times e^{-\int_\zeta^a (\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma) i_h(t-a+\sigma,\sigma))d\sigma} d\zeta + \int_0^a \delta_v(\zeta) \int_0^{\zeta} \varepsilon_v(\theta) k(t-\zeta+\theta) s_v(t-\zeta+\theta,\zeta) \times e^{-\int_\theta^\zeta (\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\theta e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta \Big\} da, \tag{23}$$

and taking the limit supremum when $t \longrightarrow \infty$, it can be easily shown that:

$$\lim_{t\longrightarrow+\infty} k(t) \leq R_0 * C * \limsup_{t\longrightarrow+\infty} k(t) \tag{24}$$

which implies that

$$\limsup_{t\longrightarrow+\infty} k(t) = 0,$$

resulting in

$$\limsup_{t\longrightarrow+\infty} i_h(t,a) = \limsup_{t\longrightarrow+\infty} i_v(t,a) = 0,$$

and

$$\limsup_{t\longrightarrow+\infty} l_h(t,a) = \limsup_{t\longrightarrow+\infty} l_v(t,a) = 0.$$

Following the fact that

$$\limsup_{t\longrightarrow+\infty} (s_h(t,a) + l_h(t,a) + i_h(t,a) + r_h(t,a)) = 1,$$

and

$$\limsup_{t\longrightarrow+\infty} (s_v(t,a) + l_v(t,a) + i_v(t,a)) = 1,$$

we have

$$\limsup_{t\longrightarrow+\infty} s_h(t,a) = 1, \limsup_{t\longrightarrow+\infty} s_v(t,a) = 1,$$

which completes the proof. □

4. Existence and Local Stability of Endemic Equilibrium

An endemic equilibrium is solution of the following time independent system:

$$\begin{cases} \frac{ds_h^*(a)}{da} = \alpha_h(a) + \gamma(a)r_h^*(a) - (\alpha_h(a) + \varepsilon_h(a)k^* - \eta(a)i_h^*(a))s_h^*(a), \\ \frac{dl_h^*(a)}{da} = \varepsilon_h(a)k^* s_h^*(a) - (\alpha_h(a) + \delta_h(a) - \eta(a)i_h^*(a))l_h^*(a), \\ \frac{di_h^*(a)}{da} = \delta_h(a)l_h^*(a) - (\alpha_h(a) + \lambda(a) + \eta(a) - \eta(a)i_h^*(a))i_h^*(a), \\ \frac{dr_h^*(a)}{da} = \lambda(a)i_h^*(a) - (\alpha_h(a) + \gamma(a) - \eta(a)i_h^*(a))r_h^*(a), \\ \frac{ds_v^*(a)}{da} = \alpha_v(a) - (\alpha_v(a) + \varepsilon_v(a)k^*(a))s_v^*(a), \\ \frac{dl_v^*(a)}{da} = \varepsilon_v(a)k^*(a)s_v^*(a) - (\alpha_v(a) + \delta_v(a))l_v^*(a), \\ \frac{di_v^*(a)}{da} = \delta_v(a)l_v^*(a) - \alpha_v(a)i_v^*(a), \\ k^* = \int_0^\omega \beta(a)(i_h^*(a) + i_v^*(a))da, \end{cases} \tag{25}$$

From system (25), we have:

$$s_h^*(a) = \int_0^a (\alpha_h(\zeta) + \gamma(\zeta)r_h^*(\zeta))e^{-\int_\zeta^a (\alpha_h(\sigma)+\varepsilon_h(\sigma)k^* - \eta(\sigma)i_h^*(\sigma))d\sigma} d\zeta, \tag{26}$$

$$l_h^*(a) = \int_0^a \varepsilon_h(\zeta)k^* s_h^*(\zeta)e^{-\int_\zeta^a (\alpha_h(\sigma)+\delta_h(\sigma) - \eta(\sigma)i_h^*(\sigma))d\sigma} d\zeta, \tag{27}$$

$$i_h^*(a) = \int_0^a \delta_h(\zeta)l_h^*(\zeta)e^{-\int_\zeta^a (\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma) - \eta(\sigma)i_h^*(\sigma))d\sigma} d\zeta, \tag{28}$$

$$r_h^*(a) = \int_0^a \lambda(\zeta)i_h^*(\zeta)e^{-\int_\zeta^a (\alpha_h(\sigma)+\gamma(\sigma) - \eta(a)i_h^*(\sigma))d\sigma} d\zeta, \tag{29}$$

$$s_v^*(a) = \int_0^a \alpha_v(\zeta)e^{-\int_\zeta^a (\alpha_v(\sigma)+\varepsilon_v(\sigma)k^*)d\sigma} d\zeta, \tag{30}$$

$$l_v^*(a) = \int_0^a \varepsilon_v(\zeta)k^* s_v^*(\zeta)e^{-\int_\zeta^a (\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\zeta, \tag{31}$$

$$i_v^*(a) = \int_0^a \delta_v(\zeta)l_v^*(\zeta)e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta, \tag{32}$$

$$k^* = \int_0^{\omega} \beta(a)(i_h^*(a) + i_v^*(a))da. \tag{33}$$

Using equations (27) and (31) in (28) and (32), respectively, we get:

$$i_h^*(a) = \int_0^a \delta_h(\zeta) \int_0^{\zeta} \varepsilon_h(\theta) k^* s_h^*(\theta) \times e^{-\int_\theta^\zeta (\alpha_h(\sigma)+\delta_h(\sigma))d\sigma} d\theta e^{-\int_\zeta^a (\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma)i_h^*(\sigma))d\sigma} d\zeta, \tag{34}$$

$$i_v^*(a) = \int_0^a \delta_v(\zeta) \int_0^{\zeta} \varepsilon_v(\theta) k^* s_v^*(\theta) e^{-\int_\theta^\zeta (\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\theta e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta. \tag{35}$$

Using relations (34) and (35) in (33), we get:

$$1 = \int_0^{\omega} \beta(a) \Bigg\{ \int_0^a \delta_h(\zeta) [\int_0^{\zeta} \varepsilon_h(\theta) s_h^*(\theta) e^{-\int_\theta^\zeta (\alpha_h(\sigma)+\delta_h(\sigma))d\sigma} d\theta] \times e^{-\int_\zeta^a (\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma)i_h^*(\sigma))d\sigma} d\zeta + \int_0^a \delta_v(\zeta) [\int_0^{\zeta} \varepsilon_v(\theta) s_v^*(\theta) e^{-\int_\theta^\zeta (\alpha_v(\sigma)+\delta_v(\sigma))d\sigma} d\theta] e^{-\int_\zeta^a \alpha_v(\sigma)d\sigma} d\zeta \Bigg\} da. \tag{36}$$

We denote the right hand side of (36) by $G(k^*)$ and we let $\int_0^{\omega} \beta(a)da := \beta^+$. It can be shown that $k^*G(k^*) < \beta^+$. For $k^* = \beta^+$, we have $G(\beta^+) < 1$. On the other hand, $G(0) > 1$. Thus, a unique real positive solution exists in the interval $(0, \beta^+)$ which corresponds to the endemic state.

To show the local asymptotic stability, let us first introduce a perturbation in system (2) at the endemic equilibrium state. By setting $s_h(t,a) = \hat{s}_h(t,a) + s_h^*(a), l_h(t,a) = \hat{l}_h(t,a) + l_h^*(a), i_h(t,a) = \hat{i}_h(t,a) + i_h^*(a), r_h(t,a) = \hat{r}_h(t,a) + r_h^*(a), s_v(t,a) = \hat{s}_v(t,a) + s_v^*(a), l_v(t,a) = \hat{l}_v(t,a) + l_v^*(a), i_v(t,a) = \hat{i}_v(t,a) + i_v^*(a),$

system (2) becomes:

$$\begin{cases}
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{s}_h(t,a)=\gamma(a)\hat{r}_h(t,a)-(\alpha_h(a)+\varepsilon_h(a)k^*-\eta(a)i_h^*(a))\hat{s}_h(t,a)\\
-(\alpha_h(a)+\varepsilon_h(a)\hat{k}(t)-\eta(a)\hat{i}_h(t,a))s_h^*(a),\\
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{l}_h(t,a)=\varepsilon_h(a)\hat{k}(t)s_h^*+\varepsilon_h(a)k^*\hat{s}_h(t,a)-(\alpha_h(a)+\delta(a)\\
-\eta(a)i_h^*(a))\hat{l}_h(t,a),\\
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{i}_h(t,a)=\delta(a)\hat{l}_h(t,a)-(\alpha_h(a)+\lambda(a)+\eta(a)-\eta(a)i_h^*(a))\hat{i}_h(t,a),\\
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{r}_h(t,a)=\lambda(a)\hat{i}_h(t,a)-(\alpha_h(a)+\gamma(a)-\eta(a)i_h^*(a))\hat{r}_h(t,a),\\
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{s}_v(t,a)=-(\alpha_v(a)+\varepsilon_v(a)k^*)\hat{s}_v(t,a)-\varepsilon_v(a)\hat{k}(t)s_v^*(a),\\
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{l}_v(t,a)=\varepsilon_v(a)\hat{k}(t)s_v^*(a)+\varepsilon_v(a)k^*\hat{s}_v(t,a)-(\alpha_v(a)+\delta_v(a))\hat{l}_v(t,a),\\
(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\hat{i}_v(t,a)=\delta_v(a)\hat{l}_v(t,a)-\alpha_v(a)\hat{i}_v(t,a),\\
\hat{k}(t)=\int_0^\omega(\hat{i}_h(t,a)+\hat{i}_v(t,a))da.
\end{cases}\tag{37}$$

We consider the solution in the exponential form as: $\hat{s}_h(t,a)=\hat{s}_h(a)e^{zt}, \hat{l}_h(t,a)=\hat{l}_h(a)e^{zt}, \hat{i}_h(t,a)=\hat{i}_h(a)e^{zt}, \hat{r}_h(t,a)=\hat{r}_h(a)e^{zt}, \hat{s}_v(t,a)=\hat{s}_v(a)e^{zt}, \hat{l}_v(t,a)=\hat{l}_v(a)e^{zt}, \hat{i}_v(t,a)=\hat{i}_v(a)e^{zt}$.

We obtain the following linear system from (37):

$$\begin{cases}
\frac{d\hat{s}_h(a)}{da}=\gamma(a)\hat{r}_h(a)-(\varepsilon_h(a)\hat{k}+\eta(a)\hat{i}_h(a))s_h^*(a)\\
-(\alpha_h(a)+\varepsilon_h(a)k^*-\eta(a)i_h^*(a)+z)\hat{s}_h(a),\\
\frac{d\hat{l}_h(a)}{da}=\varepsilon_h(a)k^*\hat{s}_h(a)+\varepsilon_h(a)\hat{k}s_h^*(a)-(\alpha_h(a)+\lambda(a)\\
-\eta(a)i_h^*(a)+z)\hat{i}_h(a)+\eta(a)\hat{i}_h(a)l_h^*(a),\\
\frac{d\hat{i}_h(a)}{da}=\delta_h(a)\hat{l}_h(a)-(\alpha_h(a)+\lambda(a)+\eta(a)-\eta(a)i_h^*(a)+z)\hat{i}_h(a)\\
+\eta(a)\hat{i}_h(a)i_h^*(a),\\
\frac{d\hat{r}_h(a)}{da}=\lambda(a)\hat{i}_h(a)-(\alpha_h(a)+\gamma(a)-\eta(a)i_h^*(a)+z)\hat{r}_h(a)+\eta(a)\hat{i}_h(a)r_h^*(a),\\
\frac{d\hat{s}_v(a)}{da}=-\varepsilon_v(a)\hat{k}s_h^*(a)-(\alpha_v(a)+\varepsilon_v(a)k^*+z)\hat{s}_v(a),\\
\frac{d\hat{l}_v(a)}{da}=\varepsilon_v(a)k^*\hat{s}_v(a)+\varepsilon_v(a)\hat{k}s_v^*(a)-(\alpha_v(a)+\delta_v(a)+z)\hat{l}_v(a),\\
\frac{d\hat{i}_v(a)}{da}=\delta_v(a)\hat{l}_v(a)-(\alpha_v(a)+z)\hat{i}_v(a),\\
\hat{k}=\int_0^\omega\beta(a)(\hat{i}_h(a)+\hat{i}_v(a))da.
\end{cases}\tag{38}$$

By assuming that $\hat{k}\neq 0$, and setting

$\tilde{s}_h(a)=\frac{\hat{s}_h(a)}{\hat{k}}, \tilde{l}_h(a)=\frac{\hat{l}_h(a)}{\hat{k}}, \tilde{i}_h(a)=\frac{\hat{i}_h(a)}{\hat{k}}, \tilde{r}_h(a)=\frac{\hat{r}_h(a)}{\hat{k}}, \tilde{s}_v(a)=\frac{\hat{s}_v(a)}{\hat{k}}, \tilde{l}_v(a)=$

$\frac{\hat{l}_v(a)}{\hat{k}}, \tilde{i}_v(a) = \frac{\hat{i}_v(a)}{\hat{k}}$, we get the following system:

$$\begin{cases} \frac{d\tilde{s}_h(a)}{da} = & \gamma(a)\tilde{r}_h(a) - (\varepsilon_h(a) - \eta(a)\tilde{i}_h(a))s_h^*(a) \\ & -(\alpha_h(a) + \varepsilon_h(a)k^* - \eta(a)i_h^*(a) + z)\tilde{s}_h(a), \\ \frac{d\tilde{l}_h(a)}{da} = & \varepsilon_h(a)k^*\tilde{s}_h(a) + \varepsilon_h(a)s_h^*(a) - (\alpha_h(a) + \lambda(a) - \eta(a)i_h^*(a) + z)\tilde{i}_h(a) \\ & +\eta(a)\tilde{i}_h(a)l_h^*(a), \\ \frac{d\tilde{i}_h(a)}{da} = & \delta_h(a)\tilde{l}_h(a) - (\alpha_h(a) + \lambda(a) + \eta(a) - \eta(a)i_h^*(a) + z)\tilde{i}_h(a) \\ & +\eta(a)\tilde{i}_h(a)i_h^*(a) \\ \frac{d\tilde{r}_h(a)}{da} = & \lambda(a)\tilde{i}_h(a) - (\alpha_h(a) + \gamma(a) - \eta(a)i_h^*(a) + z)\tilde{r}_h(a) \\ & +\eta(a)\tilde{i}_h(a)r_h^*(a), \\ \frac{d\tilde{s}_v(a)}{da} = & -\varepsilon_v(a)s_h^*(a) - (\alpha_v(a) + \varepsilon_v(a)k^* + z)\tilde{s}_v(a), \\ \frac{d\tilde{l}_v(a)}{da} = & \varepsilon_v(a)k^*\tilde{s}_v(a) + \varepsilon_v(a)s_v^*(a) - (\alpha_v(a) + \delta_v(a) + z)\tilde{l}_v(a), \\ \frac{d\tilde{i}_v(a)}{da} = & \delta_v(a)\tilde{l}_v(a) - (\alpha_v(a) + z)\tilde{i}_v(a), \\ 1 = & \int_0^\omega \beta(a)(\tilde{i}_h(a) + \tilde{i}_v(a))da. \end{cases} \tag{39}$$

We denote by $Q(z)$ the right hand side of the last equation of system (39). Integrating system (39), we get :

$$\tilde{s}_h(a) = \int_0^a \left[\gamma(\zeta)\tilde{r}_h(\zeta) - \left(\varepsilon_h(\zeta) + \eta(\zeta)\tilde{i}(\zeta)\right)s_h^*\right] e^{-\int_\zeta^a (\alpha_h(\sigma) + \varepsilon_h(\sigma)k^* - \eta(\sigma)i_h^*(\sigma) + z)d\sigma} d\zeta, \tag{40}$$

$$\tilde{l}_h(a) = \int_0^a \left[\varepsilon_h(\zeta)k^*\tilde{s}_h(\zeta) + \varepsilon_h(\zeta)s_h^*(\zeta)\right] e^{-\int_\zeta^a (\alpha_h(\sigma) + \delta_h(\sigma) - \eta(\sigma)i_h^*(\sigma) + z)d\sigma} d\zeta, \tag{41}$$

$$\tilde{i}_h(a) = \int_0^a \delta_h(\zeta)\tilde{l}_h(\zeta) e^{-\int_\zeta^a (\alpha_h(\sigma) + \lambda(\sigma) - \eta(\sigma)i_h^*(\sigma) + z)d\sigma} d\zeta, \tag{42}$$

$$\tilde{r}_h(a) = \int_0^a \lambda(\zeta)\tilde{i}_h(\zeta) e^{-\int_\zeta^a (\alpha_h(\sigma) + \gamma(\sigma) - \eta(\sigma)i_h^*(\sigma) + z)d\sigma} d\zeta, \tag{43}$$

$$\tilde{s}_v(a) = -\int_0^a \varepsilon_v(\zeta)s_v^*(\zeta) e^{-\int_\zeta^a (\alpha_v(\sigma) + \varepsilon_v(\sigma)k^* + z)d\sigma} d\zeta, \tag{44}$$

$$\tilde{l}_v(a) = \int_0^a \left[\varepsilon_v(\zeta)k^*\tilde{s}_v(\zeta) + \varepsilon_v(\zeta)s_v^*(\zeta)\right] e^{-\int_\zeta^a(\alpha_v(\sigma)+\delta_v(\sigma)+z)d\sigma} d\zeta, \tag{45}$$

$$\tilde{i}_v(a) = \int_0^a \delta_v(\zeta)\tilde{l}_v(\zeta)e^{-\int_\zeta^a(\alpha_v(\sigma)+z)d\sigma} d\zeta. \tag{46}$$

The expression of $Q(z)$ becomes:

$$\begin{aligned}
Q(z) &= \int_0^\omega \beta(a)\Big\{ \int_0^a \delta_h(\zeta) \int_0^\zeta \varepsilon_h(\theta)s_h^*(\theta)e^{-\int_\theta^\zeta \left[\alpha_h(\sigma)+\delta_h(\sigma)-\eta(\sigma)i_h^*(\sigma)+z\right]d\sigma} d\theta \\
&\times e^{-\int_\zeta^a \left[\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma)i_h^*(\sigma)+z\right]d\sigma} d\zeta + \int_0^a \delta_v(\zeta) \int_0^\zeta \varepsilon_v(\theta)s_v^*(\theta)e^{-\int_\theta^\zeta \left[\alpha_v(\sigma)+\delta_v(\sigma)+z\right]d\sigma} d\theta \\
&\times e^{-\int_\zeta^a \left[\alpha_v(\sigma)+z\right]d\sigma} d\zeta \Big\} da \\
&+ \int_0^\omega \beta(a)\Big\{ \int_0^a \delta_h(\zeta) \int_0^\zeta \left[\varepsilon_h(\theta)k^*\tilde{s}_h(\theta) + \eta(\theta)\tilde{i}_h(\theta)l_h^*(\theta)\right] e^{-\int_\theta^\zeta \left[\alpha_h(\sigma)+\delta_h(\sigma)-\eta(\sigma)i_h^*(\sigma)+z\right]d\sigma} d\theta \\
&\times e^{-\int_\zeta^a \left[\alpha_h(\sigma)+\lambda(\sigma)+\eta(\sigma)-\eta(\sigma)i_h^*(\sigma)+z\right]d\sigma} d\zeta + \int_0^a \delta_v(\zeta) \int_0^\zeta \varepsilon_v(\theta)k^*\tilde{s}_v(\theta)e^{-\int_\theta^\zeta \left[\alpha_v(\sigma)+\delta_v(\sigma)+z\right]d\sigma} d\theta \\
&\times e^{-\int_\zeta^a \left[\alpha_v(\sigma)+z\right]d\sigma} d\zeta \Big\} da.
\end{aligned} \tag{47}$$

It follows that $Q(z)$ is a decreasing function such that $\lim\limits_{z\longrightarrow+\infty} Q(z) = 0$ if the perturbations introduced at the endemic steady state are small.

Setting $z = 0$ in (47), we notice that the first integral is equal to one. From equation (44), $\tilde{s}_v < 0$ and from (40) $\tilde{s}_h$ is negative if the difference $\left[\gamma(\zeta)\tilde{r}_h(\zeta) - \left(\varepsilon_h(\zeta) + \eta(\zeta)\tilde{i}(\zeta)\right)s_h^*\right]$ is less than zero which shows that the second integral is negative if $\left[\varepsilon_h(\theta)k^*\tilde{s}_h(\theta) + \eta(\theta)\tilde{i}_h(\theta)l_h^*(\theta)\right]$ is less than zero. Thus, with the previous considerations, we have $Q(0) < 1$. It follows that a unique negative real solution of $Q(z) = 1$ exists and as consequence, we have the following theorem.

Theorem 3. *The endemic equilibrium state is locally asymptotically stable if* $R_0 > 1$.

5. Conclusion

We have considered a model including temporal immunity in the human population. In some regions like sub-saharan countries where populations are consistently exposed to biting by infectious mosquitoes, people acquire an induced

immunity which is significant in the disease dynamics. In our future work, we will extend our model to investigate the influence of migration phenomenon in the disease dynamic.

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