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ON THE LIMIT CYCLES FOR A CLASS OF FOURTH-ORDER DIFFERENTIAL EQUATIONS

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Abstract: We provide sufficient conditions for the existence of periodic solutions of the fourth-order differential equation

$$\ddddot{x} + (1+p^2)\ddot{x} + p^2 x = \epsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

where $p = p_1/p_2$ with $p_1, p_2 \in \mathbb{N}$ and p is different from $-1, 0, 1$. ϵ is a small real parameter, and F is a non-autonomous periodic function with respect to t.

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1. Introduction and Statement of the Main Results

The objective of this paper is to study the periodic solutions of the fourth-order differential equation

$$\ddddot{x} + (1+p^2)\ddot{x} + p^2 x = \epsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1}$$

where p is a rationnel different from $-1, 0, 1$, ϵ is a small real parameter and

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F is non-autonomous periodic function with respect to t. The dot denotes derivative with respect to an independent variable t.
Equations (1) appear in many places. For instance, Champneys [5] analyzes a class of equations (1) looking mainly for harmonic orbits.
When $F = \pm x^3$, then equation (1) is called the Extended Fisher-Kolmogorov equation or the Swift-Hohenberg equation see [3, 7], and in other places, see for instance the book [10] and [1, 4].
Some results on the periodic orbits for extended Fischer-Kolmogorov and Swift-Hohenberg equations of the form

$$\ddot{\ddot{x}} + q\ddot{x} + \alpha(t)x = f(t, x, \dot{x}, \ddot{x}), \tag{2}$$

with α and f functions has been studied in [4] and in the references quoted there.
The differential equation (1) when F does not depend on t has been studied in [8].
Here for the differential equation (1) we provide an analytical algorithm for studying their periodic orbits, see Theorem 1. Moreover, we shall illustrate the use of this algorithm in Corollaries 1 and 2.
We recall that a simple zero x_0^* of a real function $F(x_0)$ is defined by $F(x_0^*) = 0$ and $(\frac{dF}{dx_0})(x_0^*) \neq 0$.
Our main result on the periodic solutions of this fourth-order differential equation 1 is the following.

Theorem 1. *Let $p = p_1/p_2$ be a rationnel different from $-1, 0, 1$ with p_1 and p_2 comprime. For every $(X_0^*, Y_0^*, Z_0^*, V_0^*)$ solution of the system*

$$F_1(X_0, Y_0, Z_0, V_0) = 0, F_2(X_0, Y_0, Z_0, V_0) = 0$$
$$F_3(X_0, Y_0, Z_0, V_0) = 0, F_4(X_0, Y_0, Z_0, V_0) = 0,$$

where

$$\det\left(\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(X_0, Y_0, Z_0, V_0)}|_{(X_0,Y_0,Z_0,V_O)=(X_0^*,Y_0^*,Z_0^*,V_0^*)}\right) \neq 0, \tag{3}$$

and

$$F_1(X_0,Y_0,Z_0,V_0) = \int_0^{2\pi p_2} \cos t F(t,\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})dt$$
$$F_2(X_0,Y_0,Z_0,V_0) = -\int_0^{2\pi p_2} \sin t F(t,\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})dt$$
$$F_3(X_0,Y_0,Z_0,V_0) = \frac{-1}{p}\int_0^{2\pi p_2} \sin(pt) F(t,\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})dt$$
$$F_4(X_0,Y_0,Z_0,V_0) = \frac{-1}{p}\int_0^{2\pi p_2} \cos(pt) F(t,\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})dt,$$

with

$$\mathcal{A}(t) = \frac{X_0\sin t + Y_0\cos t - Z_0\cos(pt) + V_0\sin(pt)}{P^2-1}$$
$$\mathcal{B}(t) = \frac{X_0\cos t - Y_0\sin t + P(Z_0\sin(pt) + V_0\cos(pt))}{P^2-1}$$
$$\mathcal{C}(t) = \frac{-X_0\sin t - Y_0\cos t + P^2(Z_0\cos(pt) - V_0\sin(pt))}{P^2-1}$$
$$\mathcal{D}(t) = \frac{-X_0\cos t + Y_0\sin t - P^3(Z_0\sin(pt) + V_0\cos(pt))}{P^2-1},$$

the differential equation (1) *has a periodic solution* $x(t,\epsilon)$ *tending to the solution*

$$x(t) = \frac{X_0^*\sin t + Y_0^*\cos t - Z_0^*\cos(pt) + V_0^*\sin(pt)}{P^2-1} \tag{4}$$

of the equation

$$\ddddot{x} + (1+p^2)\ddot{x} + p^2x = 0,$$

when $\epsilon \to 0$. *Note that this solution is periodic of period* $2\pi p_2$.

Two applications of Theorem 1 for studying the periodic solutions of equation (1) are given in the following two corollaries.

Corollary 1. *If* $F(t,x,\dot{x},\ddot{x},\dddot{x}) = (x^2 + x - \dot{x})\cos(\frac{t}{2})$, *then the differential equation* (1) *with* $p = \frac{3}{2}$ *has one periodic solution* $x(t,\epsilon)$ *tending to the periodic solution* $x(t)$ *of period* 4π *given by* (4) *with* $(X_0^*, Y_0^*, Z_0^*, V_0^*) = (0, -6-3, 0)$, *of the equation* $\ddddot{x} + \frac{13}{4}\ddot{x} + \frac{9}{4}x = 0$ *when* $\epsilon \to 0$.

Corollary 2. *If* $F(t,x,\dot{x},\ddot{x},\dddot{x}) = x\dot{x} - \dot{x} + \sin(t)$, *then the differential equation* (1) *with* $p = 2$ *has three periodic solutions* $x_k(t,\epsilon)$ *for* $k = 1,2,3$ *tending to the periodic solutions* $x(t)$ *of perid* 2π *given by* (4) *with* $(X_0^*, Y_0^*, Z_0^*, V_0^*)$

equal to

$$(0,-6,-3,0),$$

$$(0,3\sqrt{5}+3,-\frac{9}{2}-\frac{3}{2}\sqrt{5},0),$$

$$(0,-3\sqrt{5}+3,-\frac{9}{2}+\frac{3}{2}\sqrt{5},0),$$

of the equation $\dddot{\dot{x}} + 5\ddot{x} + 4x = 0$ *when* $\epsilon \to 0$.

2. Basic Results on Averaging Theory

In this section we present a basic result from the averaging theory that we shall need for proving the main results of this paper. We consider the problem of bifurcation of T-periodic solutions from differential equations of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \epsilon F_1(t,\mathbf{x}) + \epsilon^2 F_2(t,\mathbf{x},\epsilon), \tag{5}$$

with $\epsilon = 0$ to $\epsilon \neq 0$ sufficiently small. Here the function $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\epsilon_0,\epsilon_0) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}), \tag{6}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. Let $\mathbf{x}(t,\mathbf{z},\epsilon)$ be the solution of the system (6) such that $\mathbf{x}(0,\mathbf{z},\epsilon) = \mathbf{z}$. We write the linearization of the unperturbed system a long a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{7}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fondamental matrix of the linear differential system (7).
We assume that there exists an open set V with $CI(V) \subset \Omega$ such that for each $\mathbf{z} \in CI(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T-periodic, where $\mathbf{x}(t,\mathbf{z},0)$ denotes the solution of the unperturbed system (6) with $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. The set $CI(V)$ is isochronous for the system (5); ie, it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $CI(V)$ is given in the following result.

Theorem 2. *(Perturbation of an isochronous set) We assume that there exists an open and bounded set V with $CI(V) \subset \Omega$ such that for each $\mathbf{z} \in CI(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T-periodic, then we consider the function $\mathcal{F} : CI(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}))dt. \tag{8}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/d\mathbf{z})(a)) \neq 0$, then there exists a T-periodic solution $\varphi(t, \epsilon)$ of system (5) such that $\varphi(0, \epsilon) \to a$ as $\epsilon \to 0$.

For an easy proof of Theorem 2 see corollary 1 of [2].

3. Proof of Theorem 1

Introducing the variables $(t, x, y, z, v) = (t, x, \dot{x}, \ddot{x}, \dddot{x})$ we write the fourth-order differential equation (1) as the following first-order differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= v, \\ \dot{v} &= -p^2 x - (1 + p^2) z + \epsilon F(t, x, y, z, v), \end{aligned} \tag{9}$$

defined in an open set Ω of $\mathbb{R}$. Of course as before the dot denotes derivative with respect to the independent variable t. system (9) with $\epsilon = 0$ will be called the unperturbed system, otherwise we have the perturbed system. The unperturbed system has a unique singular point at the origine with eigenvalues $\pm i$ and $\pm ip$. We shall write system (9) in such a way that the linear part at the origin will be in its real Jordan normal form. Then doing the change of variables $(x, y, z, v) \to (X, Y, Z, V)$ given by

$$\begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix} = \begin{pmatrix} 0 & p^2 & 0 & 1 \\ p^2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1/p & 0 & -1/p \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix} \tag{10}$$

The differential system (9) becomes

$$\begin{aligned} \dot{X} &= -Y + \epsilon G(t, X, Y, Z, V), \\ \dot{Y} &= X, \\ \dot{Z} &= -pV, \\ \dot{V} &= pZ - \frac{\epsilon}{p} G(t, X, Y, Z, V), \end{aligned} \tag{11}$$

where $G(t, X(t), Y(t), Z(t), V(t)) = F(t, A(t), B(t), C(t), D(t))$ with

$$A(t) = \frac{Y - Z}{p^2 - 1},\ B(t) = \frac{pV + X}{p^2 - 1},\ C(t) = \frac{p^2 Z - Y}{p^2 - 1},\ D(t) = \frac{p^3 V + X}{1 - p^2}.$$

Note that the linear part of the differential system (11) at the origin is in its real normal jordan form and the change of variables (10) is defined when p is different from $-1, 0, 1$, because the determinant of the matrix of the change is $(p^2 - 1)^2/p$. Now we shall apply Theorem 2 to the differential system (11) taking

$$\begin{aligned}
\mathbf{x} &= (X, Y, Z, V), t = t \\
F_0(t, \mathbf{x}) &= (-Y, X, -pV, pZ), \\
F_1(t, \mathbf{x}) &= (F(t, A(t), B(t), C(t), D(t)), 0, 0, \frac{-1}{p} F(t, A(t), B(t), C(t), D(t))), \\
F_2(t, \mathbf{x}, \epsilon) &= 0, \\
\Omega &= \mathbb{R}^4.
\end{aligned}$$

System (11) with $\epsilon = 0$ has a linear center at the origin. We remark that all linear centers are isochronous. Using the notation of Theorem 2, the periodic solution $\mathbf{x}(t, \mathbf{z})$ of this center with $\mathbf{z} = (X_0, Y_0, Z_0, V_0)$ is

$$\begin{aligned}
X(t) &= X_0 \cos t - Y_0 \sin t, \\
Y(t) &= Y_0 \cos t + X_0 \sin t, \\
Z(t) &= Z_0 \cos(pt) - V_0 \sin(pt), \\
V(t) &= V_0 \cos(pt) + Z_0 \sin(pt),
\end{aligned} \tag{12}$$

This set of periodic orbits has dimension four, all having the same period $T = 2\pi p_2$. We must calculate the zeros $\alpha = (X_0^*, Y_0^*, Z_0^*, V_0^*)$ of the system $\mathcal{F}(\alpha) = 0$, where $\mathcal{F}(\alpha)$ is geven by (8). The fundamental matrix $M(t)$ of the differential system (11) with $\epsilon = 0$ along any periodic solution is

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos(pt) & -\sin(pt) \\ 0 & 0 & \sin(pt) & \cos(pt) \end{pmatrix}$$

The inverse matrix of$M(t)$ is

$$M^{-1}(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos(pt) & \sin(pt) \\ 0 & 0 & -\sin(pt) & \cos(pt) \end{pmatrix}$$

Now computing the function $\mathcal{F}(\alpha)$ given in (8) we got that the system $\mathcal{F}(\alpha) = 0$ can be written as

$$\begin{pmatrix} F_1(X_0, Y_0, Z_0, V_0) \\ F_2(X_0, Y_0, Z_0, V_0) \\ F_3(X_0, Y_0, Z_0, V_0) \\ F_4(X_0, Y_0, Z_0, V_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

where

$$\begin{aligned} F_1(X_0, Y_0, Z_0, V_0) &= \int_0^{2\pi p_2} \cos t F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{C}) dt, \\ F_2(X_0, Y_0, Z_0, V_0) &= -\int_0^{2\pi p_2} \sin t F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) dt, \\ F_3(X_0, Y_0, Z_0, V_0) &= \frac{-1}{p} \int_0^{2\pi p_2} \sin(pt) F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) dt, \\ F_4(X_0, Y_0, Z_0, V_O) &= \frac{-1}{p} \int_0^{2\pi p_2} \cos(pt) F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) dt, \end{aligned}$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ as in statement of Theorem 1.
The zeros $(X_0^*, Y_0^*, Z_0^*, V_0^*)$ of system (13) with respect to the variables X_0, Y_0, Z_0 and V_0 provide periodic orbits of system (13) with $\epsilon = 0$ sufficiently small if they are simple i.e if

$$\det\left(\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(X_0, Y_0, Z_0, V_0)}\Big|_{(X,Y,Z,V)=(X_0^*, Y_0^*, Z_0^*, V_0^*)} \right) \neq 0$$

Going back through the change of variables, for every simple zero $(X_0^*, Y_0^*, Z_0^*, V_0^*)$ of system (13), we obtain $2\pi p_2$ periodic solution $x(t)$ of differential system(1) for $\epsilon \neq 0$ sufficiently small such that $x(t)$ tends to periodic solution

$$x(t) = \frac{X_0^* \sin t + Y_0^* \cos t - Z_0^* \cos(pt) + V_0^* \sin(pt)}{p^2 - 1}$$

of equation

$$\ddddot{x} + (1 + p^2)\ddot{x} + p^2 x = 0,$$

where $\epsilon \to 0$. Hence Theorem 1 is proved.

4. Proof of Corollaries 1 and 2

Proof of Corollary 1. We have the equation

$$\dddot{x} + \frac{13}{4}\ddot{x} + \frac{9}{4}x = \epsilon(x^2 + x - \dot{x})\cos(\frac{t}{2}) \tag{14}$$

which corresponds to the case $p = \frac{3}{2}$ and $F(t, x, \dot{x}, \ddot{x}, \dddot{x}) = (x^2 + x - \dot{x})\cos(\frac{t}{2})$. The functions F_1 F_2, F_3 and F_4 of Theorem 1 are

$$\begin{aligned}
F_1(X_0, Y_0, Z_0, V_0) &= \pi\left(\frac{-4}{5}Z_0 - \frac{6}{5}V_0 + \frac{16}{25}X_0V_0 - \frac{16}{25}Y_0Z_0\right),\\
F_2(X_0, Y_0, Z_0, V_0) &= \pi\left(\frac{6}{5}Z_0 - \frac{4}{5}V_0 - \frac{16}{25}X_0Z_0 - \frac{16}{25}Y_0V_0\right)\\
F_3(X_0, Y_0, Z_0, V_0) &= \pi\left(\frac{-8}{15}X_0 - \frac{8}{15}Y_0 - \frac{32}{75}X_0Y_0\right),\\
F_4(X_0, Y_0, Z_0, V_O) &= \pi\left(\frac{-16}{75}Y_0^2 + \frac{16}{75}X_0^2 + \frac{8}{15}X_0 - \frac{8}{15}Y_0\right),
\end{aligned}.$$

System $F_1 = F_2 = F_3 = F_4 = 0$ has only one real solution

$$(X_0^*, Y_0^*, Z_0^*, V_0^*) = (\frac{-5}{2}, \frac{-5}{2}, 0, 0).$$

Since the Jacobian

$$\det\left(\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(X_0, Y_0, Z_0, V_0)}|_{(X_0,Y_0,Z_0,V_0=(\frac{-5}{2},\frac{-5}{2},0,0)}\right) = 469.9187726 \neq 0.$$

Therefore, from Theorem 1 equation (14) has one periodic solution given by eq(4).

Proof of Corollary 2. We have the equation

$$\dddot{x} + 5\ddot{x} + 4x = \epsilon(x\dot{x} - \dot{x} + \sin(t)) \tag{15}$$

which corresponding to the case $p = 2$ and $F(t, x, \dot{x}, \ddot{x}, \dddot{x}) = x\dot{x} - \dot{x} + \sin(t)$. The functions F_1 F_2, F_3 and F_4 of Theorem 1 are

$$\begin{aligned}
F_1(X_0, Y_0, Z_0, V_0) &= \pi(\frac{1}{18}Y_0V_0 + \frac{1}{18}X_0Z_0 - \frac{1}{3}X_0),\\
F_2(X_0, Y_0, Z_0, V_0) &= \pi(\frac{1}{18}X_0V_0 - \frac{1}{18}Y_0Z_0 - \frac{1}{3}Y_0 - 1),\\
F_3(X_0, Y_0, Z_0, V_0) &= \pi(\frac{1}{36}Y_0^2 - \frac{1}{36}X_0^2 + \frac{1}{3}Z_0),\\
F_4(X_0, Y_0, Z_0, V_O) &= \pi(\frac{-1}{18}X_0Y_0 + \frac{1}{3}V_0).
\end{aligned}$$

System $F_1 = F_2 = F_3 = F_4 = 0$ has three solutions $(X_0^*, Y_0^*, Z_0^*, V_0^*)$ given by

$$s_1 = (0, -6, -3, 0), \quad s_2 = (0, 3\sqrt{5} + 3, \frac{-9}{2} - \frac{3}{2}\sqrt{5}, 0),$$

$$s_3 = (0, -3\sqrt{5} + 3, \frac{-9}{2} + \frac{3}{2}\sqrt{5}, 0).$$

Since the Jacobian

$$\det\left(\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(X_0, Y_0, Z_0, V_0)}|_{(X_0,Y_0,Z_0,V_0)=X_0^*,Y_0^*,Z_0^*,V_0^*)}\right)$$

for these three solutions $(X_0^*, Y_0^*, Z_0^*, V_0^*)$ is $-0.3006453428, 1.087745072,$ 0.4154816443 respectively we obtain using Theorem 1 three solutions given by (4).

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