MONOTONIC INTEGRABLE SOLUTION FOR A MIXED TYPE INTEGRAL AND DIFFERENTIAL INCLUSIONS OF FRACTIONAL ORDERS

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ABSTRACT: In this paper, we present a global existence theorem of positive monotonic integrable solution for the mixed type integral inclusion of fractional order

\[ x(t) \in p(t) + \int_0^1 k(t, s) F_1(s, I^{\beta} f_2(s, x(s))) ds, \quad t \in [0, 1], \quad \beta > 0. \]

The initial value problem of arbitrary (fractional) orders differential inclusion will be considered as an application.

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1. INTRODUCTION

Let \( \alpha \in (0, 1) \), and consider the fractional order functional integral inclusion

\[ x(t) \in p(t) + F_1(t, I^{\alpha} f_2(t, x(\varphi(t))), \quad t \in [0, 1]. \]
In [8], the authors proved the existence of global integrable solution for the nonlinear functional integral inclusion (1), where the set-valued map \( F_1 : (0, 1) \times \mathbb{R}^+ \to 2^{\mathbb{R}^+} \) has nonempty closed values which is satisfy Caratheodory and growth conditions.

The existence of positive monotonic continuous solution of the mixed type integral inclusion

\[
x(t) \in p(t) + \int_0^1 k(t, s) F_1(s, \int_0^s f_2(s, x(s))ds, \quad t \in [0, 1], \quad \beta > 0.
\]  

was proved in [9] by using Arzela -Ascoli Theorem and applying Schauder’s fixed-point Theorem, also, in [7] the local existence of monotonic integrable solution of the mixed type nonlinear integral equation of fractional order

\[
x(t) = p(t) + \int_0^1 k(t, s) f_1(s, \int_0^s f_2(s, x(s))ds, \quad t \in [0, 1], \quad \beta > 0
\]

has been proved in when the two functions \( f_1 \) and \( f_2 \) are monotonic nondecreasing (in both variables).

Here, we omit this conditions and prove a global existence theorem for a positive nondecreasing integrable solution of (3).

As a generalization of our results we study the global existence of positive monotonic integrable solution for the integral inclusion (2), and the initial value problem of arbitrary (fractional) orders differential inclusion

\[
\frac{dx}{dt} \in p(t) + \int_0^1 k(t, s) F_1(s, D^\alpha x(s))ds, \text{ a.e. } t \in [0, 1], \quad \beta > 0
\]

\[
x(0) = x_0
\]

will be studied.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let \( L^1 = L^1(I) \) be the class of Lebesgue integrable function on the interval \( I = [a, b] \), where \( 0 \leq a < b < \infty \) and let \( \Gamma(.) \) be the gamma function.

**Definition 1.** The fractional integral of the function \( f(.) \in L^1(I) \) of order \( \alpha \in \mathbb{R}^+ \) is defined by (cf. [11] [12] and [14] [16])

\[
I^\alpha \ f(t) = \int_a^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s) \ ds.
\]

For the properties of the fractional order integral see [11], [10].
Definition 2. The (Caputo) fractional derivative $D_α^g$ of order $α ∈ (a, b]$ of the absolutely continuous function $g$ is defined as (see [2] [12] [13] and [16])

$$D_α^g g(t) = I_{a}^{1-α} \frac{d}{dt} g(t) , \quad t ∈ [a, b]$$

Recall that the operator $T$ is compact if it is continuous and maps bounded sets into relatively compact sets from a subspace $U ⊂ X$ into the Banach space $X$ is denoted by $C(U, X)$.

Moreover, let us define the operator $T$ as

$$T x(t) = p(t) + \int_{0}^{1} k(t, s)f_1(s, I^β f_2(s, x(s))) ds,$$

The following two theorems will be needed in the proof of our main result.

Theorem 3. (Nonlinear Alternative of Leray-Shauder type) [4]. Let $U$ be an open subset of a convex set $D$ in a Banach space $X$. Assume $0 ∈ U$ and $T ∈ C(\bar{U}, D)$. Then either:

$(A_1)$ $T$ has a fixed point in $\bar{U}$, or

$(A_2)$ there exists $γ ∈ (0, 1)$ and $x ∈ ∂U$ such that $x = γTx$.

Theorem 4. (Kolmogorov Compactness Criterion) [5]. Let $Ω ⊆ L^p(0, 1), 1 ≤ P ≤ ∞$. If:

(i) $Ω$ is bounded in $L^p(0, 1)$, and

(ii) $x_h → x$ as $h → 0$ uniformly with respect to $x ∈ Ω$,

then $Ω$ is relatively compact in $L^p(0, 1)$, where

$$x_{h}(t) = \frac{1}{h} \int_{0}^{t+h} x(s) \, ds.$$ 

3. MAIN RESULTS

To facilitate our discussion, let us first state the following assumptions:

(i) $p ∈ L^1$;
(ii) \( f_i : [0,1] \times R^+ \to R^+ \), \( i = 1, 2 \) satisfies caratheodory condition i.e \( f_i \) are measurable in \( t \) for any \( x \in R^+ \) and continuous in \( x \) for almost all \( t \in [0,1] \).

There exists two functions \( t \to a(t) , t \to b(t) \) such that
\[
|f_i(t,x)| \leq a_i(t) + b_i(t)|x|, \quad i = 1, 2 \quad \forall t \in [0,1] \quad \text{and} \quad x \in R,
\]
where \( a_i(.) \in L^1 \) and \( b_i(.) \) are measurable and bounded;

(iii) \( k : [0,1] \times R^+ \to R^+ \) is measurable and the integral operator \( K \) defined by
\[
(Kx)(t) = \int_0^1 k(t,s)x(s)ds
\]
map \( L^1 \) into itself and such that
\[
\int_0^1 |k(t,s)|dt < k;
\]

(iv) Assume that every solution \( x(.) \in L^1 \) to the equation
\[
x(t) = \gamma(p(t) + \int_0^1 k(t,s)f_1(s, I^\beta f_2(s, x(s)))ds), \quad t \in [0,1], \quad 0 < \beta < 1, \quad \gamma \in (0,1),
\]
satisfies \( \|x\| \neq r \) ( \( r > 0 \) is arbitrary but fixed).

Now, we are in position to state and prove our main result

**Theorem 5.** Let the assumptions (i)-(iv) be satisfied. Then equation (3) has at least one solution in \( L^1 \).

**Proof.** Let \( x \) be an arbitrary element in the open set \( B_r = \{ x : \|x\| < r , \ r > 0 \} \).

Then from assumption (i) and (ii) we have,
\[
\|Tx\| = \int_0^1 |(Tx)(t)| \ dt
\]
\[
\leq \int_0^1 |p(t)|dt + \int_0^1 |k(t,s)||f_1(s, \int_0^s \frac{(s - \tau)^{\beta - 1}}{\Gamma(\beta)} f_2(\tau, x(\tau))d\tau)|ds
\]
\[
\leq \|p\| + k \int_0^1 |a_1(s) + b_1(s)| \int_0^s \frac{(s - \tau)^{\beta - 1}}{\Gamma(\beta)} f_2(\tau, x(\tau))d\tau|ds
\]
\[
\leq \|p\| + k \|a_1\|ds + k \int_0^1 |b_1(s)| \int_0^s \frac{(s - \tau)^{\beta - 1}}{\Gamma(\beta)} |f_2(\tau, x(\tau))|d\tau ds
\]
\[
\leq \|p\| + k\|a_1\| + kb_1 \int_0^1 \int_0^s \frac{(s - \tau)^{\beta - 1}}{\Gamma(\beta)}[a_2(\tau) + b_2(\tau)|x(\tau)|]d\tau ds
\]
\[
\leq \|p\| + k\|a_1\|
Hence, the above inequality means that the operator $T$ maps $B_r$ into $L^1$.

Now, we will show that $T$ is compact. To achieve this goal we will apply Theorem 3. So let $\Omega$ be a bounded subset of $B_r$. Then $T(\Omega)$ is bounded in $L^1$ i.e condition (i) of Theorem 4 satisfied.

It remains to show that $(Tx)_h \rightarrow Tx$ in $L^1$ as $h \rightarrow 0$ uniformly with respect to $Tx \in \Omega$. We have the following:

$$
\| (Tx)_h - (Tx) \| = \int_0^1 |(Tx)_h(t) - (Tx)(t)| dt,
$$

$$
= \int_0^1 \frac{1}{h} \int_t^{t+h} (Tx)_h(\tau) d\tau - (Tx)(t) dt,
$$

$$
= \int_0^1 \frac{1}{h} \int_t^{t+h} ((Tx)_h(\tau) - (Tx)(t)) d\tau dt
$$

$$
\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |p(\tau) - p(t)| d\tau dt
$$

$$
+ \int_0^1 \frac{1}{h} \int_t^{t+h} \left| \int_0^1 k(\tau, s) f_1(s, I^\beta f_2(s, x(s))) ds \right| d\tau dt
$$

$$
- \int_0^1 k(t, s) f_1(s, I^\beta f_2(s, x(s))) ds |d\tau dt
$$

$$
\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |p(\tau) - p(t)| d\tau dt
$$

$$
+ \int_0^1 \frac{1}{h} \int_t^{t+h} \left| \int_0^1 (k(\tau, s) - k(t, s)) f_1(s, I^\beta f_2(s, x(s))) ds \right| d\tau dt,
$$

$$
\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |p(\tau) - p(t)| d\tau dt
$$

$$
+ \int_0^1 \frac{1}{h} \int_t^{t+h} \int_0^1 |k(\tau, s) - k(t, s)| f_1(s, I^\beta f_2(s, x(s))) ds d\tau dt.
$$
since \( f_1, f_2 \in L^1 \), we get \( I^\beta f_2 \in L^1 \) and \( K f_1 f_2 \in L^1 \), then
\[
\frac{1}{h} \int_t^{t+h} \int_0^1 |k(\tau, s) - k(t, s)| f_1(s, I^\beta f_2(s, x(s))) ds d\tau \to 0
\]
Moreover, \( p(.) \in L^1 \). So, we have
\[
\frac{1}{h} \int_t^{t+h} |p(\tau) - p(t)| d\tau \to 0
\]
for a.e \( t \in L^1 \). Therefore, by Theorem 4 we have that \( T(\Omega) \) is relatively compact, that is, \( T \) is compact operator.

Set \( U = B_r \) and \( D = X = L^1[0,1] \). Then in the view of assumption (iv) condition \((A_2)\) of Theorem 3 does not hold. Theorem 1, implies that \( T \) has a fixed point. This completes the proof. \( \square \)

**Corollary 6.** Let the assumptions of Theorem 5 be satisfied. If the function \( p \) is nondecreasing on \([0,1]\) and the kernel \( k \) is nondecreasing with respect to \( t \in [0,1] \) then the solution of equation (3) is nondecreasing.

**Proof.** For \( t_1, t_2 \in [0,1] \) and \( t_1 < t_2 \), we have
\[
x(t_1) = p(t_1) + \int_0^1 k(t_1, s) f_1(s, I^\beta f_2(s, x(s))) ds,
\]
\[
\leq p(t_2) + \int_0^1 k(t_2, s) f_1(s, I^\beta f_2(s, x(s))) ds = x(t_2).
\]
\( \square \)

### 4. Mixed Type Integral Inclusion

Consider now inclusion (2), where \( F_1 : [0,1] \times R^+ \to 2^{R^+} \) has nonempty closed values.

As an important consequence of the Theorem 5 we can present the following:

**Theorem 7.** Let the functions \( f_1, p \) and \( k \) be satisfied all Assumptions of Theorem 5.

Let the set-valued function \( F_1 \) be satisfied the following:

(i) \( F_1(t, x) \) are non empty, closed and convex for all \((t, x) \in [0,1] \times R^+ \),

(ii) \( F_1(t,.) \) is lower semicontinuous from \( R^+ \) into \( R^+ \),

(iii) \( F_1(.,.) \) is measurable,
(iv) There exists a function $a \in L^1$ and a positive number $b$ such that

$$|F_1(t, x)| \leq a_1(t) + b_1(t)|x| \forall t \in [0, 1].$$

Then there exists at least one positive non-decreasing solution $x \in L^1$ of the integral inclusion (2).

**Proof.** By conditions (i) − (iv) (see [1], [3], [6] [15]), we can find a selection function (Caratheodory selection) $f_1 : [0, 1] \times R^+ \rightarrow R^+$ such that $f_1(t, x) \in F_1(t, x)$ for all $(t, x) \in [0, 1] \times R^+$. which satisfies the assumption (2) of Theorem 5.

Clearly all assumptions of Theorem 5 are hold, then there exists at least one positive solution $x \in L^1$ such that

$$x(t) - p(t) = \int_0^1 K(t, s) f_1(s, I^\alpha f_2(s, x(s))) \in \int_0^1 K(t, s) F_1(s, I^\beta f_2(s, x(s))).$$

Now, we can easily proved the following corollary

**Corollary 8.** Let the assumptions of Theorem 7 and Corollary 6 be satisfied, then the solution of integral inclusion (2) is non-decreasing.

### 5. Differential Inclusion

Consider now the initial value problem of the differential inclusion (4) with the initial data (5).

**Theorem 9.** Let the assumptions of Theorem 5 be satisfied, then the initial value problem (4)-(5) has at least one positive non-decreasing solution $x \in L^1$.

**Proof.** Let $y(t) = \frac{dx(t)}{dt}$, then equation (4) transformed to the integral inclusion

$$y(t) \in P(t) + \int_0^1 k(t, s) F_1(s, I^{1-\alpha} y(s)) ds$$

which by Theorem 7 has at least one positive solution $y \in L^1$.

This implies that the existence of absolutely continuous solution

$$x(t) = x_0 + \int_0^t y(s) ds$$

is a positive non-decreasing solution of the initial-valued problem (4)-(5).
REFERENCES


