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COMMENTS ON SOME INVERTED CUMULATIVE DISTRIBUTIONS: "SATURATION IN THE HAUSDORFF SENSE", APPLICATIONS

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ABSTRACT: In this note we study properties of some inverted cumulative distribution functions (CDFs). More precisely, we prove estimates for the "saturation" – d about Hausdorff metric using two–parameters generalized inverted exponential c.d.f. The technique used can be successfully applied to other commonly used CDFs in practice. We consider also modified families of adaptive functions with "polynomial variable transfer" with applications to the Antenna–feeder Analysis. Numerical examples, illustrating our results using CAS MATHEMATICA are given.

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Key Words: generalized inverted exponential c.d.f., inverse Weibull c.d.f., "adaptive function of Gumbel–type with polynomial variable transfer", Hausdorff distance, filter characteristic, emitting chart

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1. INTRODUCTION

The generalized inverted exponential distribution is popular for modeling lifetime data in engineering, reliability, biomedical sciences and life testing [1]–[4].

Definition 1. We consider the following two–parameters generalized inverted exponential cumulative distribution function:

$$F(t) = 1 - \left(1 - e^{-\frac{\delta}{t}}\right)^a \tag{1}$$

for $t > 0$, $\delta > 0$, $a > 1$.

Various modifications of this "powerful" class of functions (see, Section 2.3) have been proposed and studied by a number of researchers.

Definition 2. [5, 6] The Hausdorff distance (the H-distance) [5] $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\Big\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B|| \Big\}, \tag{2}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In this article we study some properties of the family (1) and prove estimate for the "saturation" – d about Hausdorff metric. The technique used can be successfully applied to other commonly used CDFs in practice (see, Section 2.3). We consider also modified families of adaptive functions with "polynomial variable transfer" with applications to the Antenna–feeder Analysis.

2. MAIN RESULTS

2.1. "SATURATION" IN THE HAUSDORFF SENSE BY THE C.D.F. (1)

1. For the "saturation" – d in the Hausdorff sense to the horizontal asymptote using $F(t)$ we have

$$F(d) = 1 - d, \tag{3}$$

i.e. d is the solution of the nonlinear equation

$$e^{\frac{\delta}{d}} - \frac{1}{1 - d^{\frac{1}{a}}} = 0.$$

Let

$$K(a,d) = \frac{-d^{\frac{1}{a}}}{1-d^{\frac{1}{a}}} := K$$

and

$$G(d) = e^{\frac{\delta}{d}} + K\frac{1}{d^{\frac{1}{a}}} = 0. \tag{4}$$

Consider the following good approximation to the $G(d)$ (see, Figure 1 b).

$$H(d) = e^{\frac{\delta}{d}+d} - K\ln\delta^a = 0. \tag{5}$$

From (5) we find

$$\frac{\delta}{d} + d = \ln(K\ln\delta^a).$$

For sufficiently small values of δ we have

$$d \approx \frac{\delta}{\ln(K\ln\delta^a)}.$$

Let $d < 1/8$. Then $K \approx -\frac{1}{7}$.

Thus, we prove the following

Theorem 3. *For sufficiently small values of $\delta > 0$ and $d \leq \frac{1}{8}$ for the "saturation" – d we have*

$$d \approx \frac{\delta}{\ln\ln\frac{1}{(\delta^a)^{\frac{1}{7}}}}. \tag{6}$$

For other approximation results, see [7]–[9].

2.2. NUMERICAL EXPERIMENTS. SOME APPLICATIONS

Let $\delta = 0.01$ and $a = 2$. From the nonlinear equation (4) we find $d = 0.0430197$. From (6) for the "saturation" we have $d \approx 0.0364409$. The family $F(t)$ is plotted on Figure 1 a. In the case $\delta = 0.01$; $a = 3$, see Figure 2.

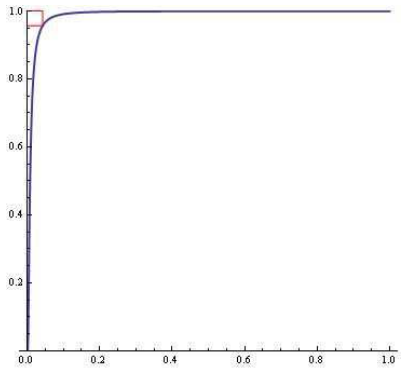


Figure 1. a) $\delta = 0.01$; $a = 2$; Hausdorff distance (from nonlinear equation (4)):
$d = 0.0430197$

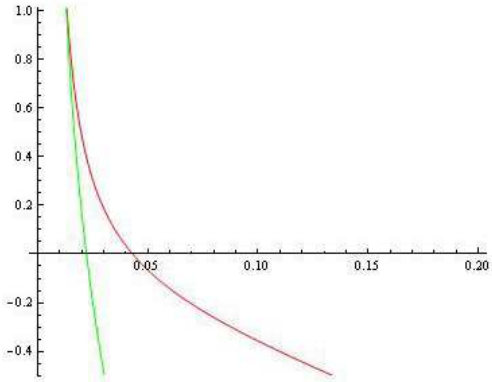
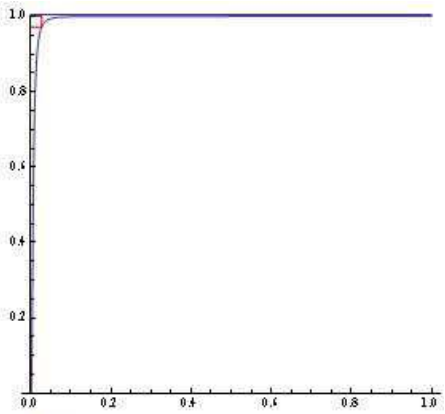


Figure 1. b) The functions $G(d)$ and $H(d)$



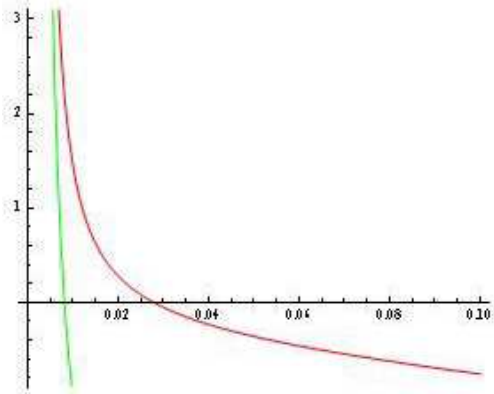


Figure 2. The case $\delta = 0.01$; $a = 3$.

Some computational examples using relations (4) and (6) are presented in Table 1.

δ	a	d computed by (4)	d computed by (6)
0.01	2	0.0430197	0.0364409
0.001	3	0.027736	0.0147084
0.005	4	0.0123349	0.00451356
0.005	6	0.00835236	0.00330417

Table 1. Bounds for d computed by (4) and (6) for various values of δ and a.

Consider the following model with "polynomial variable transfer":

$$\begin{aligned} F_n^*(t) &= 1 - \left(1 - e^{-\frac{\delta}{|f(t)|}}\right)^a \\ f(t) &= \sum_{i=0}^{n} a_i t^i, \quad a_0 = 0. \end{aligned} \tag{7}$$

For example, let

$$F_3^*(t) = 1 - \left(1 - e^{-\frac{\delta}{|t(1-t)(2-t)(3-t)|}}\right)^a.$$

A typical "filter characteristic" by using model $F_3^*(t)$ for $\delta = 2.4$, $a = 3.5$ is plotted on Figure 3.

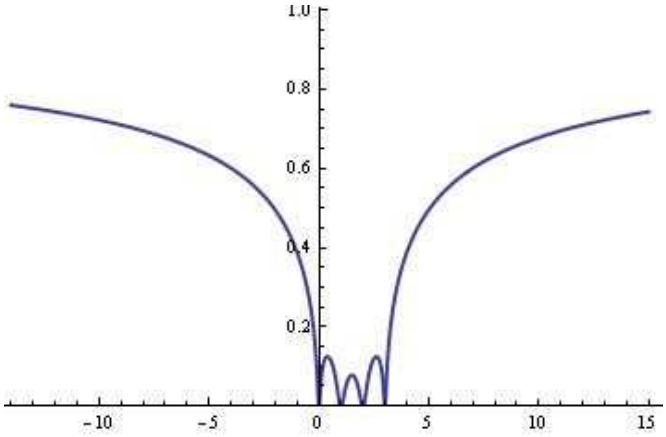


Figure 3. A typical "filter characteristic" by using model $F_3^*(t)$ for $\delta = 2.4$, $a = 3.5$.

Let $t = b\cos\theta + c$. Consider the function $|F_6^*(t)|$. Then, for example, typical "emitting chart" using $|F_6^*(t)|$ for

$$n = 6,\ \delta = 0.22,\ a = 1.1,\ a_0 = 0,\ a_1 = -0.1,\ a_2 = 1.1,\ a_3 = -1.1,$$
$$a_4 = 0.15,\ a_5 = 0.5,\ a_6 = -0.02,\ b = -1.2,\ c = 0.001.$$

is plotted on Figure 4.

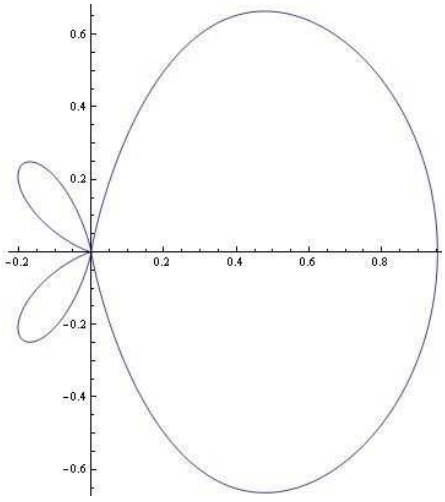


Figure 4. A typical "emitting chart" using $|F_6^*(t)|$.

2.3. A LOOK AT SOME INVERTED CUMULATIVE DISTRIBUTION FUNCTIONS (CDF). SOME APPLICATIONS

In [9] we study some properties of an new inverse Weibull cumulative function proposed by Afify, Shawky and Nassar [10]

$$F_1(t) = 1 - \frac{\ln\left(1+\delta-\delta^{e^{-\frac{\delta}{t}}}\right)}{\ln\delta} \tag{8}$$

and prove that for sufficiently small values of δ and $d \leq \frac{1}{2}$ the following estimate for the "saturation" – d about Hausdorff metric is valid:

$$d \approx \frac{\delta}{1+\ln\left(\ln\left(\frac{1}{\delta}\right)\right)}. \tag{9}$$

The following "inverted" (CDFs) are widely used for data analysis from various branches of science:

$$F_2(t) = 1 - \ln\left(1+e-e^{e^{-\frac{\delta}{t}}}\right); \tag{10}$$

$$F_3(t) = \frac{1-\left(1-e^{-\frac{\delta}{t}}\right)^a}{1+\left(1-e^{-\frac{\delta}{t}}\right)^a}; \tag{11}$$

$$F_4(t) = 1 - \left(1-e^{-\left(\frac{\delta}{t}\right)^a}\right)^b \tag{12}$$

$$F_5(t) = \frac{1-\left(1-e^{-\left(\frac{\delta}{t}\right)^a}\right)^b}{1+\left(1-e^{-\left(\frac{\delta}{t}\right)^a}\right)^b} \tag{13}$$

$$F_6(t) = \frac{e^{\alpha(1+\lambda t^{-\phi})^{-2\eta}}-1}{e^{\alpha}-1} \tag{14}$$

$$F_7(t) = \left(1-\left(1-e^{-\frac{\delta}{t^b}}\right)^l\right)^m \tag{15}$$

$$F_8(t) = 1 - \left(1-\left(\frac{\alpha^{e^{-\frac{\lambda}{t}}}-1}{\alpha-1}\right)^{\phi}\right)^b. \tag{16}$$

The reader can formulate the corresponding approximation problem for CDFs following results from Theorem 3 and general ideas given in [9].

Consider the following "adaptive function of Gumbel–type (15) with polynomial variable transfer":

$$G_7(t) = A\left(1-\left(1-e^{-\delta f(t)^{-b}}\right)^l\right)^m, \tag{17}$$

where

$$f(t) = \sum_{i=0}^{n} a_i t^i; \quad a_0 = 0.$$

Let $t = r\cos\theta + c$. The simulation using $|G_7(\theta)|$ for a) $A = 1.33$; $\delta = 2.95$; $b = 0.15$; $l = 1.5$; $m = 0.3$; $r = 1.59$; $c = -0.39$ for fixed $f(t) = t(1-t)(0.7-t)(0.5-t)$ is given on Figure 5.

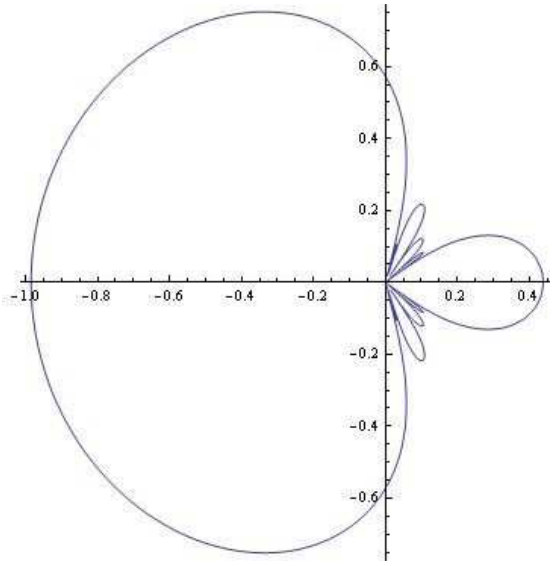


Figure 5. The function $|G_7(\theta)|$ in the case a).

The simulation using $|G_7(\theta)|$ for b) $A = 2.65$; $\delta = 2.9$; $b = 0.1$; $l = 1.7$; $m = 0.5$; $r = -1.002$; $c = 0.15$ for fixed $f(t) = t(0.8-t)(0.9-t)(0.5-t)(0.1-t)$ is depicted on Figure 6.

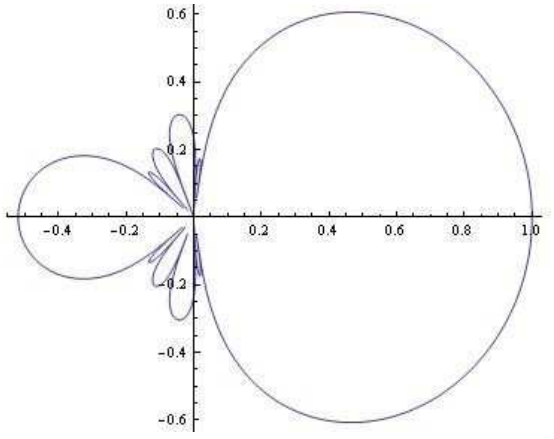


Figure 6. The function $|G_7(\theta)|$ in the case b).

Evidently, the "adaptive function" $|G_7(\theta)|$ is of the form of "antenna pattern".

Of course, the question of the practical realization of the activation functions which are generated as emitting charts remains open.

Remark 4. We bring to the reader's attention the following interesting modification of the model (8)

$$F_1(t) = 1 - \frac{\ln\left(1+\delta-\delta^{e^{-\left(\frac{\delta}{t}\right)^{\delta}}}\right)}{\ln\delta}, \tag{18}$$

which can be considered as an adaptive function, in the light of the research conducted in the article.

Remark 5. In the next lemma we present one technical result for the basic model $F(t)$ (1).

Lemma 6. *The following inequality holds*

$$F_0(t) \le F(t) \le F_{00}(t),$$

where

$$F_0(t) = 1 - \left(\frac{\delta}{t}\right)^{\alpha} \quad \text{and} \quad F_{00}(t) = 1 - \left(\frac{\delta}{\delta+t}\right)^{\alpha}. \tag{19}$$

Proof. The proof follows immediately from the following well know inequalities

$$\frac{x}{x+1} \le 1 - e^{-x} \le x$$

that holds true for every $x > -1$. Figure 7 is a graphical representation of Lemma 6. Note that functions F_0 and F_{00} can be used as "confidential bounds".

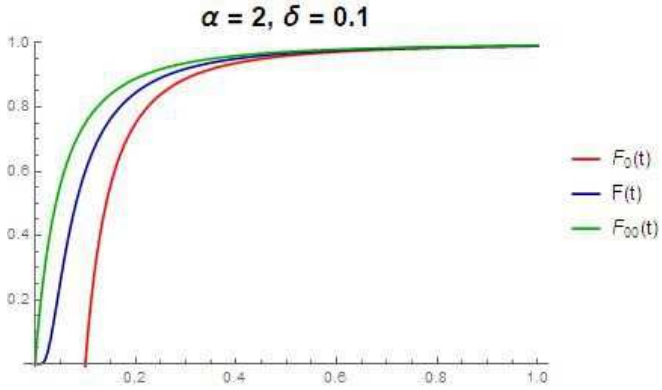


Figure 7. "Confidential bounds".

The characteristics "confidential bounds" and "saturation in the Hausdorff sense" – d are important for researchers in choosing an appropriate model for approximating specific data from different branches of scientific knowledge.

For some modelling and approximation problems, see [11]–[20]. □

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