International Journal of Differential Equations and Applications
Volume **21**, No. 2 (2022), pages: 109-120
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

DYNAMICS OF THE LIENARD POLYNOMIAL SYSTEM USING DICKSON POLYNOMIALS OF THE $(m+1)$-TH KIND. THE LEVEL CURVES

Vesselin Kyurkchiev[1], Anton Iliev[1,2], Asen Rahnev[1]
and Nikolay Kyurkchiev[1,2]

[1]Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

[2] Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, BULGARIA

ABSTRACT: In this article a model with Dickson polynomials of the $(m+1)$-th kind $D_{n,m}(x,a) := \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{n-mi}{n-i} \left(\begin{array}{c} n-i \\ i \end{array} \right) (-a)^i x^{n-2i}$ as corrections in the Lienard differential system is presented. The level curves are studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, Dickson polynomials of the $(m+1)$-th kind as "correcting factors" in the Lienard system, level curves

Received: December 1, 2022 **Revised:** December 27, 2022
Published: December 28, 2022 **doi:** 10.12732/ijdea.v21i2.9
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

125 yeras ago, L. Dickson [1] introduced a class of polynomials in the context of investigating polynomials that induce permutations of finite fields. These polynomials are called *Dickson polynomials* (for more details see [2]–[5]).

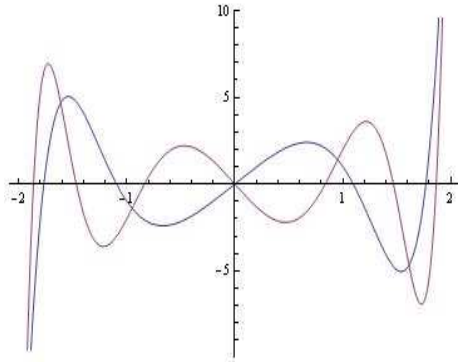

Figure 1: Dickson polynomials for fixed $a = 1$; $m = 4$ and $n = 7$; $n = 9$.

Our previous results were related to the study of the class of Lienard polynomial systems of the type

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & Poly_i(x) + \epsilon f_i(x) y \end{array} \right. \tag{1}$$

where $0 \leq \epsilon < 1$; $f_i(x)$ are specially chosen polynomials, and $Poly_i(x)$ are some of the known Dickson polynomials of the first, second and third kind [7]–[8].

In this article a model with Dickson polynomials of the $(m+1)$-th kind as corrections in the Lienard system is presented. The level curves are studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. MAIN RESULTS

Research can be successfully continued using other known classes of polynomials as corrections to the classical Lienard differential model. The Dickson polynomials of the $(m+1)$–th kind are given by (see for example [6])

$$D_{n,m}(x,a) := \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{n-mi}{n-i} \left(\begin{array}{c} n-i \\ i \end{array} \right) (-a)^i x^{n-2i}; \ a \in R.$$

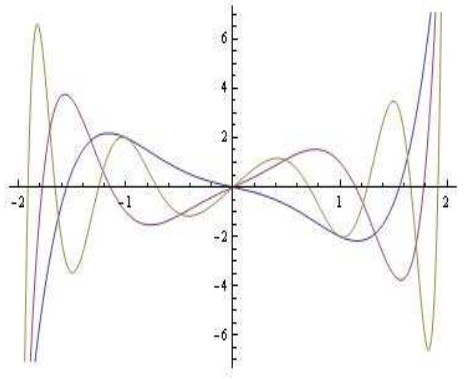

Figure 2: Dickson polynomials $D_{n,3}(x,1)$ for $n = 5, 7, 11$.

For example

$$\begin{cases} D_{7,4}(x,1) &= x^7 - 3x^5 - 2x^3 + 5x \\ D_{9,4}(x,1) &= x^9 - 5x^7 + 3x^5 + 10x^3 - 7x \\ D_{5,3}(x,a) &= x^5 - 2ax^3 - a^2x \\ D_{7,3}(x,a) &= x^7 - 4ax^5 + 2a^2x^3 + 2a^3x \\ D_{9,3}(x,a) &= x^9 - 6ax^7 + 9a^2x^5 - 3a^4x \\ D_{11,3}(x,a) &= x^{11} - 8ax^9 + 20a^2x^7 - 14a^3x^5 - 5a^4x^3 + 4a^5x \end{cases}$$

Dickson polynomials for fixed $a = 1$; $m = 4$ and $n = 7$; $n = 9$ are depicted in Fig. 1. Dickson polynomials $D_{n,3}(x,1)$ for $n = 5, 7, 11$ are depicted in Fig. 2.

Let $Poly_i(x)$ (in (1)) coincides with Dickson polynomials $D_{n,m}(x,1)$. Without going into details, we will note some more interesting level curves:

The case 1): $n = 7$; $m = 4$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{1}{8}x^8 + \frac{1}{2}x^6 + \frac{1}{2}x^4 - \frac{5}{2}x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 3.

The case 2): $n = 9$; $m = 4$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{1}{10}x^{10} + \frac{5}{8}x^8 - \frac{1}{2}x^6 - \frac{5}{2}x^4 + \frac{7}{2}x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 4.

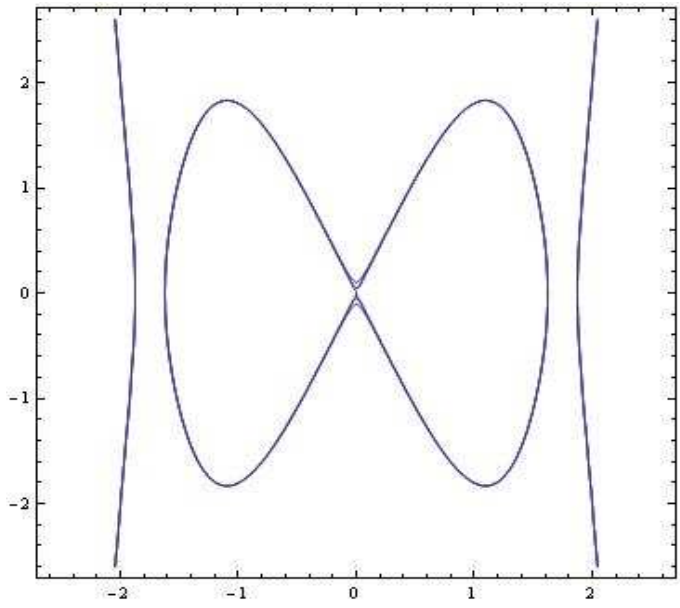

Figure 3: Level curves (the case 1).

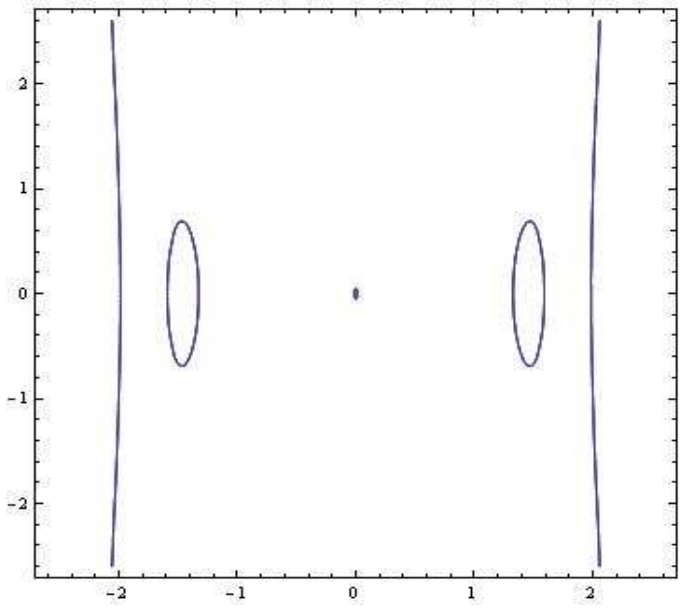

Figure 4: Level curves (the case 2).

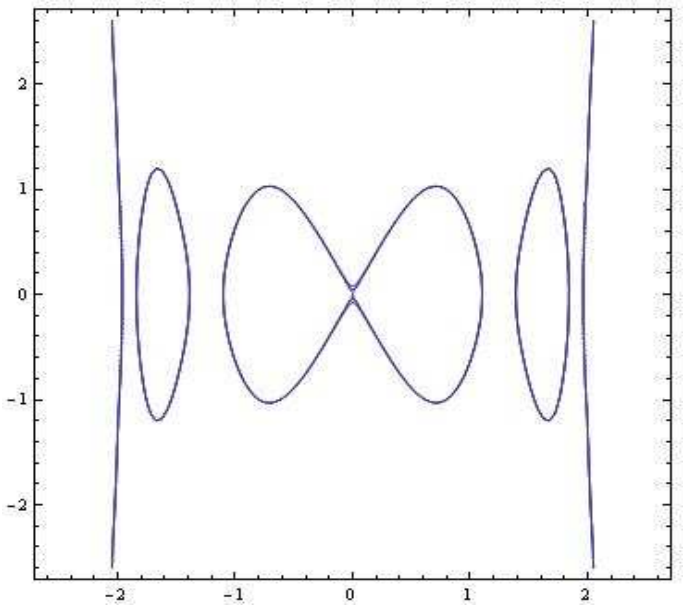

Figure 5: Level curves (the case 3).

The case 3): $n = 11$*;* $m = 3$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{1}{12}x^{12} + \frac{4}{5}x^{10} - \frac{5}{2}x^8 + \frac{7}{3}x^6 + \frac{5}{4}x^4 - 2x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 5.

Some simulations

The simulations on the Lienard–type system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -D_{7,4}(x,1) + \epsilon f(x)y \end{cases} \tag{2}$$

where $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ are depicted on Fig. 6.

The simulations on the Lienard–type system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -D_{9,4}(x,1) + \epsilon f(x)y \end{cases} \tag{3}$$

where $f(x) = x - \frac{1}{3}x^3$ are depicted on Fig. 7.

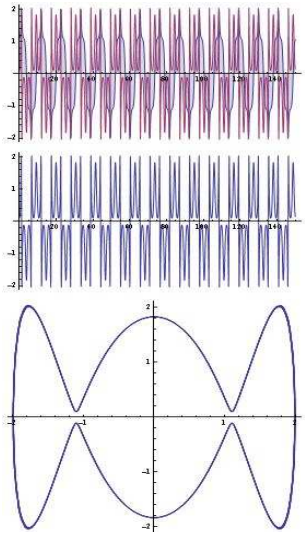

Figure 6: The simulations (system (2)) for $x_0 = 1.1; y_0 = 0.1; \epsilon = 0.0001$).

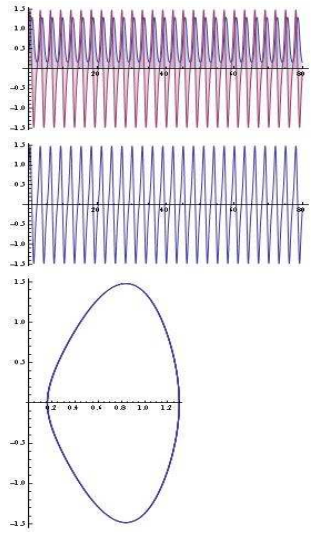

Figure 7: The simulations (system (3)) for $x_0 = 0.4; y_0 = 0.9; \epsilon = 0.0001$).

```
Print["Model using Dickson polynomial of the (m+1)-th kind"];

m = Input["Input m"]; (* 3 *)
Print["m = ", m];
Print["x'[t] =y[t]"];

Print["y'[t] =-(x[t]^7-3x[t]^5-2x[t]^3+5x[t])+
0.0001(x[t]-x[t]^3+x[t]^5-1/7*x[t]^7)*y[t]"];
x0 = Input["Input initial condition - x[0]"]; (* 1.1 *)
Print["Initial condition x0 = ", x0];
y0 = Input["Input initial condition - y[0]"]; (* 0.1 *)
Print["Initial condition y0 = ", y0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];
Print["Graphics of the solutions of the system of differential equations"];

NDSolve[{x'[t] == y[t], y'[t] == -(x[t]^7 - 3 x[t]^5 - 2 x[t]^3 + 5 x[t]) +
      0.0001 * (x[t] - x[t]^3 + x[t]^5 - 1/7 * x[t]^7) * y[t],
   x[0] == x0, y[0] == y0}, {x, y}, {t, t0, t1}];

Plot[Evaluate[{x[t], y[t]} /. First[%]], {t, t0, t1}, Filling → Axis]
exactsol2 = NDSolve[{x'[t] == y[t], y'[t] == -(x[t]^7 - 3 x[t]^5 - 2 x[t]^3 + 5 x[t]) +
       0.0001 * (x[t] - x[t]^3 + x[t]^5 - 1/7 * x[t]^7) * y[t],
    x[0] == x0, y[0] == y0}, {x, y}, {t, t0, t1}];

data = Table[{x[t], y[t]} /. exactsol2[[1]], {t, t0, t1}];

Plot[Evaluate[y[t] /. exactsol2], {t, t0, t1}]
ParametricPlot[Evaluate[{x[t], y[t]} /. exactsol2], {t, t0, t1}]
```

Figure 8: Example module in *CAS Mathematica*.

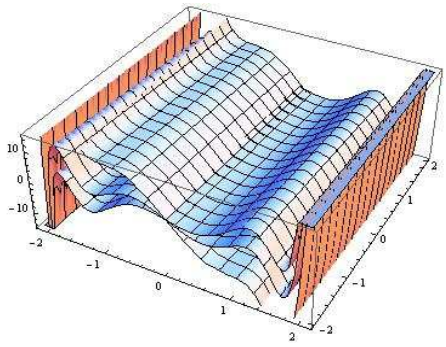

Figure 9: The catastrophe surfaces in the light of Zeeman considerations.

Example module in *CAS Mathematica* is given in Fig. 8.

Consider the following model in the light of Zeeman's approach:

$$\begin{cases} \dfrac{dx}{dt} &= c(F(x)-y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{4}$$

with $c>0$ and

$$F(x) = -px + 10x^3 + 3x^5 - 5x^7 + x^9$$

The catastrophe surfaces $(x,y,p) = F(x) - y$ $(p = 5, 10, 20)$ for the model is depicted on Fig. 9.

Polynomials of this type can be used also as correction factors in some differential models.

For other results see [9]–[21], [23].

Applications.

It is easy to take into account that the change of the variable t with $t = b\cos\theta + c$ (θ is the azimuthal angle and c is the phase difference) in the $y(t)$-component of the solution of the systems (2) and (3) leads to radiation diagrams.

A) The simulation on the system (3) (for $x_0 = 1.15; y_0 = 0.15; b = 0.7; c = 1.4$ in interval $(-\pi, \pi)$) is depicted on Fig. 10.

B) The simulation on the system (4) (for $x_0 = 0.4; y_0 = 0.9; b = 0.6; c = 1.2$ in interval $(-\pi, \pi)$) is depicted on Fig. 11.

Acknowledgment

This work has been accomplished with the financial support by the Grant No BG05M2OP001-1.001-0003, financed by the Science and Education for Smart Growth

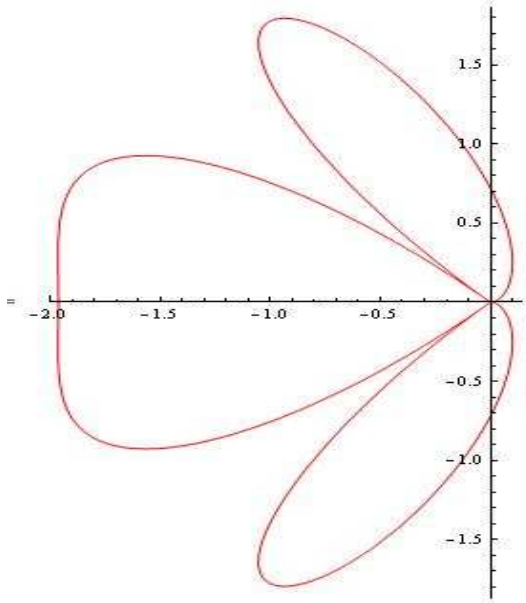

Figure 10: A typical emitting chart (Example A.)

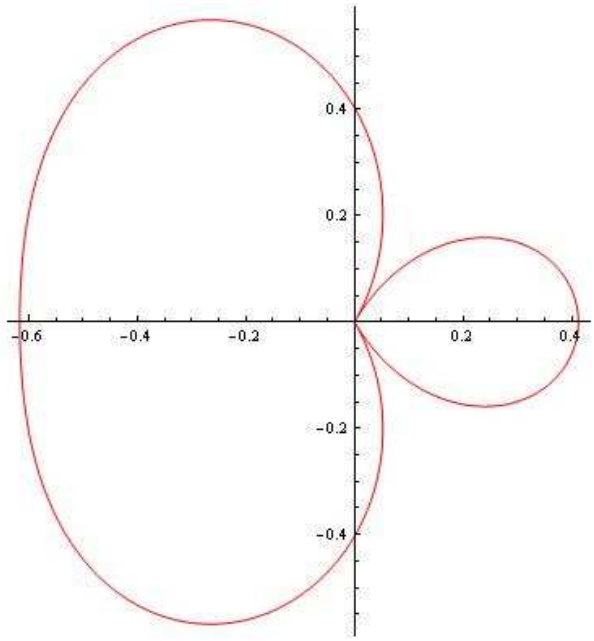

Figure 11: A typical emitting chart (Example B.)

Operational Program (2014-2020) and co-financed by the European Union through the European structural and Investment funds.

REFERENCES

[1] L. E. Dickson, The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group I, II, *Ann. of Math.* (1897).

[2] R. Lidl, G. L. Mullen, G. Turnwald, Dickson polynomials, Longman Scientific and Technical, Essex, United Kingdom, 1993.

[3] T. Stoll, Complete decomposition of Dickson-type polynomials and related Diophantine equations, *Journal of Number Theory*, 128 (2008), 1157–1181.

[4] N. Fernando, S. Manukure, A note on Dickson polynomials of the third kind and Legendre functions, *J. of Inequalities and Special Functions*, 10 (1), 2019, 151–160.

[5] M. Cipu, Dickson polynomials that are permutations, *Serdica Math. J.*, 30 (2004), 177–194.

[6] Q. Wang, J. Yucas, Dickson polynomials over finite fields, *Finite Fields Appl.*, 18 (2013), 814–831.

[7] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Simulations on the Lienard polynomial system with Dickson-type polynomial corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 31–44.

[8] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, A note on the extended Lienard system with Dickson polynomials of the third kind as corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 73–87.

[9] V. Kyurkchiev, N. Kyurkchiev, On an extended relaxation oscillator model: number of limit cycles, simulations. I, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[10] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, A technique for simulating the dynamics of some extended relaxation oscillator models. II, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[11] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Another extended polynomial Lienard systems: simulations and applications. III, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No. 1 (2022), 55–65.

[12] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Investigations on some polynomial Lienard-type systems: number of limit cycles, simulations, *International Journal of Differential Equations and Applications*, **21**, No. 1 (2022), 117–126.

[13] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, On some extended oscillator models: a technique for simulating and studying their dynamics, Plovdiv, Plovdiv University Press (2022); ISBN 978-619-7663-13-6.

[14] N. Kyurkchiev, A. Iliev, On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms* 15, 12, 2022, ISSN:1999-4893; https://doi.org /10.3390/a15120462.

[15] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Lienard system with first kind Chebyshev's polynomial–correction in the light of Melnikov's approach. Simulations and possible applications, Proc. of the Int. Conf. "Informatics, Mathematics, Education and their Application" (IMEA'2022), Pamporovo 2022 (accepted).

[16] E. Angelova, V. Arnaudova, T. Terzieva, A. Malinova, Investigations on a differential system with correction of Zernike–type radial polynomials. Simulations, Proc. of the Int. Conf. "Informatics, Mathematics, Education and their Application" (IMEA'2022), Pamporovo 2022 (accepted).

[17] N. Kyurkchiev, The effects on the dynamics of Lienard equation with Morse–type corrections: level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022).

[18] A. Malinova, T. Terzieva, O. Rahneva, E. Angelova, Legendre polynomials as "correction factors" in the Lienard differential system. Simulations, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[19] A. Golev, V. Arnaudova, Lienard system with "correcting factors" of the type of interpolating polynomials of some basic functions, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No 1 (2022)

[20] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Gegenbauer polynomials as correction in the Lienard planar system: Melnikov's approach, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 45–57.

[21] A. Golev, V. Arnaudova, Lienard system with "correcting factors" of the type of interpolating polynomials of some basic functions, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No 1 (2022), 67–80.

[22] N. Kyurkchiev, A. Andreev, V. Popov, Iterative methods for the computation of

all multiple roots of an algebraic polynomial. *Annuaire Univ. Sofia, Fac. Math. Mech.* **78** (1984), 178–185.

[23] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, On the Lienard System with Some "Corrections of Polynomial–type": Number of Limit Cycles, Simulations and Possible Applications. Part II., Plovdiv, Plovdiv University Press (2022), ISBN: 978-619-7663-41-9.