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APPLICATION OF FOURIER METHOD TO INVESTIGATION OF THE DIRICHLET PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract: This paper proves the existence and uniqueness of classical boundary value problem solutions for partial differential equation with a discreet deviating argument. The research is based on the variable separation method. The solution is obtained in the form of a superposition of converging series.

Key Words: boundary value problem, partial differential equation, divergent argument, the eigenvalue and eigenfunction problem

1. Introduction

In recent years, equations with deviating argument have been attracting more and more attention. This applies to both equations with distributed delay [8, 11, 10], and to equations with discrete deviation [1, 4, 5, 7]. On the one hand, such interest can be explained by the theoretical novelty of such research and its relation to other relevant areas (such as equations with fractional derivatives [3]), but on the other hand - if may be due to the significant practical application if presents (e.g. [6, 9]). However, despite the recent progress achieved in research of such equations the theory of boundary value problems for partial differential equations with discrete delay is far from being complete.

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This paper sets the criteria for unique solvability of boundary value problems in partial differential equations with a delay argument in the rectangular region on the basis of the variable separation method.

2. Statement of the Problem

Consider the equation

$$Lu \equiv \alpha u_{xx}(x,t) + \beta u_{tt}(x,t) + \gamma u(x, t_0 - t) = 0, \tag{1}$$

in the rectangular region $\Omega = \{(x,t) | \, 0 < x < x_0, \; 0 < t < t_0\}$, where $\alpha, \beta, \gamma, x_0, t_0$ – are prescribed constants. For equation (1) in region Ω is researched

Problem 1. Find a regular solution $u(x,t)$ to equation (1) class

$$C\left(\overline{\Omega}\right) \bigcap C^{2,4}_{x,t}(\Omega),$$

which satisfies the conditions

$$\begin{array}{ll} u(0,t) = \varphi_1(t), & u(x_0,t) = \varphi_2(t), \\ u(x,0) = \varphi_3(x), & u(x,t_0) = \varphi_4(x), \end{array} \tag{2}$$

where $\varphi_i \; \left(i = \overline{1,4}\right)$ – are prescribed and continuously differentiable functions, where $\varphi_1(0) = \varphi_3(0)$, $\varphi_1(t_0) = \varphi_4(0)$, $\varphi_2(0) = \varphi_3(x_0)$, $\varphi_2(t_0) = \varphi_4(x_0)$.

3. Proof of Existence and Uniqueness of Problem

For problem 1 the following theorem is true:

Theorem 1. *Assume*
1) $\varphi_1(t), \varphi_2(t) \in C^4[0,t_0]$, $\varphi_3(x), \varphi_4(x) \in C^4[0,x_0]$, *where* $x_0 \in \left(0, \frac{\pi}{2\sqrt{2}}\right)$, $t_0 \in \left(0, \frac{\pi\sqrt{\beta}}{\sqrt{2\gamma}}\right)$;
2) $\varphi_1^{(k)}(0) = \varphi_1^{(k)}(t_0)$, $\varphi_2^{(k)}(0) = \varphi_2^{(k)}(t_0)$, $\varphi_3^{(k)}(0) = \varphi_3^{(k)}(x_0)$, $\varphi_4^{(k)}(0) = \varphi_4^{(k)}(x_0), k = \overline{0,4}$;
3) $\int\limits_0^{t_0} \varphi_1(t)\,dt = 0, \int\limits_0^{t_0} \varphi_2(t)\,dt = 0$;
4) $\gamma > 0$, $\alpha < -\frac{\gamma x_0^2}{\pi^2 n^2}$, $\beta > \frac{2\gamma t_0^2}{(\pi + 2\pi n)^2}$, $n \in \mathrm{N}$.
Then problem 1 is solvable in the required class of functions.

To prove theorem 1 we decompose problem 1 into two auxiliary problems in region:

$$Lu_1 \equiv 0, \tag{3}$$

$$u_1(x,0) = 0, u_1(x,t_0) = 0, \tag{4}$$

$$u_1(0,t) = \varphi_1(t), u_1(x_0,t) = \varphi_2(t), \tag{5}$$

$$Lu_2 \equiv 0, \tag{6}$$

$$u_2(0,t) = 0, u_2(x_0,t) = 0, \tag{7}$$

$$u_2(x,0) = \varphi_3(x), u_2(x,t_0) = \varphi_4(x), \tag{8}$$

where $u_1 + u_2 = u$.

We will seek a solution of equation (3), which satisfies all conditions (4) and (5) in the form of

$$u_1 = X_1(x) \cdot T_1(t). \tag{9}$$

Substituting (9) in (3) and omitting inferior indices for brevity sake, we obtain

$$\frac{\beta T''(t) + \gamma T(t_0 - t)}{T(t)} = -\frac{\alpha X''(x)}{X(x)} = -\lambda, \tag{10}$$

where $\lambda = const$.

From here, with regards for (4) we obtain

$$\beta T''(t) + \gamma T(t_0 - t) + \lambda T(t) = 0, \tag{11}$$

$$T(0) = T(t_0) = 0. \tag{12}$$

We investigate the problem (11), (12). Differentiate (11), twice and obtain the relation:

$$\beta T^{IV}(t) + \gamma T''(t_0 - t) + \lambda T''(t) = 0. \tag{13}$$

On the other hand, from (11) we obtain:

$$T''(t_0 - t) = -\frac{\gamma}{\beta}T(t) - \frac{\lambda}{\beta}T(t_0 - t), T(t_0 - t) = -\frac{\beta}{\gamma}T''(t) - \frac{\lambda}{\gamma}T(t). \tag{14}$$

On the basis of (11) and taking into consideration (14), we obtain

$$\beta T^{IV}(t) + 2\lambda T''(t) + \frac{\lambda^2 - \gamma^2}{\beta}T(t) = 0. \tag{15}$$

Secular equation corresponding to (15), will be as follows:

$$\beta\, r^4 + 2\lambda\, r^2 + \frac{\lambda^2 - \gamma^2}{\beta} = 0.$$

Solving the bi-quadrating equation we find $r_1 = \sqrt{\frac{\gamma-\lambda}{\beta}}$, $r_2 = -\sqrt{\frac{\gamma-\lambda}{\beta}}$, $r_3 = \sqrt{\frac{-\gamma-\lambda}{\beta}}$, $r_4 = -\sqrt{\frac{-\gamma-\lambda}{\beta}}$.

Hence, the general solution to equation (15) can be expressed as follows:

$$T(t) = C_1 e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t} + C_2 e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t} + C_3 e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t} + C_4 e^{-\sqrt{\frac{-\gamma-\lambda}{\beta}}t}. \qquad (16)$$

Investigating representation (16) for different values λ.

Case 1: $\lambda < -\gamma$. In this case (16) will be as follows:

$$T(t) = C_1 e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t} + C_2 e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t} + C_3 e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t} + C_4 e^{-\sqrt{\frac{-\gamma-\lambda}{\beta}}t}.$$

Substituting the latter equality in (11), we obtain

$$e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t}\left(C_1 + C_2 e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) + e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t}\left(C_2 + C_1 e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) +$$
$$+e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t}\left(-C_3 + C_4 e^{-\sqrt{\frac{-\gamma-\lambda}{\beta}}t_0}\right) + e^{-\sqrt{\frac{-\gamma-\lambda}{\beta}}t}\left(-C_4 + C_3 e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t_0}\right) = 0.$$

This relation is true if and only if $C_2 = -C_1 e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}$, $C_4 = C_3 e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t_0}$. In this case (16) will be as follows

$$T(t) = C_1\left(e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t} - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}(t_0-t)}\right) + C_3\left(e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t} + e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}(t_0-t)}\right).$$

Applying conditions (12), we obtain

$$\begin{cases} C_1\left(1 - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) + C_3\left(1 + e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t_0}\right) = 0, \\ -C_1\left(1 - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) + C_3\left(1 + e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t_0}\right) = 0. \end{cases}$$

System determinant

$$\Delta = 2\left(1 - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right)\left(1 + e^{\sqrt{\frac{-\gamma-\lambda}{\beta}}t_0}\right)$$

vanishes if $\lambda = \gamma$, which contradicts the assumption $\lambda < -\gamma$. Hence, $C_1 = C_3 = 0$. From this we conclude that $T(t) \equiv 0$.

Case 2: $\lambda = -\gamma$ is investigated in the same way as in the previous case. Indeed, (16) becomes:

$$T(t) = C_1 e^{\sqrt{\frac{2\gamma}{\beta}}t} + C_2 e^{-\sqrt{\frac{2\gamma}{\beta}}t} + C_3 t + C_4.$$

From here, taking into consideration (11), we obtain:

$$e^{\sqrt{\frac{2\gamma}{\beta}}t}\left(C_1 + C_2 e^{-\sqrt{\frac{2\gamma}{\beta}}t_0}\right) + e^{-\sqrt{\frac{2\gamma}{\beta}}t}\left(C_2 + C_1 e^{\sqrt{\frac{2\gamma}{\beta}}t_0}\right) + C_3(t_0 - 2t) = 0.$$

The obtained relation will be true if $C_2 = -C_1 e^{\sqrt{\frac{2\gamma}{\beta}}t_0}$, $C_3 = 0$. Then expression (16), may be expressed as follows:

$$T(t) = C_1\left(e^{\sqrt{\frac{2\gamma}{\beta}}t} - e^{\sqrt{\frac{2\gamma}{\beta}}(t_0 - t)}\right) + C_4.$$

Satisfying the last condition (12), we obtain

$$\begin{cases} C_1\left(1 - e^{\sqrt{\frac{2\gamma}{\beta}}t_0}\right) + C_4 = 0, \\ -C_1\left(1 - e^{\sqrt{\frac{2\gamma}{\beta}}t_0}\right) + C_4 = 0. \end{cases}$$

Hence, $C_1 = C_4 = 0$ and $T(t) \equiv 0$.

Case 3: $-\gamma < \lambda < \gamma$. In this case (16) may be expressed as follows:

$$T(t) = C_1 e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t} + C_2 e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t} + C_3 \cos\left(t\sqrt{\frac{\gamma+\lambda}{\beta}}\right) + C_4 \sin\left(t\sqrt{\frac{\gamma+\lambda}{\beta}}\right).$$

From here, taking into consideration (11), we obtain:

$$e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t}\left(C_1 + C_2 e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) + e^{-\sqrt{\frac{\gamma-\lambda}{\beta}}t}\left(C_2 + C_1 e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) +$$

$$+2\left[C_3 \sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)\sin\left(\frac{2t - t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right) -\right.$$

$$\left. - C_4 \cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)\sin\left(\frac{2t - t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)\right] = 0.$$

Expressing C_2 and C_4 in terms of C_1 and C_3, we express the expression for $T(t)$ as follows:

$$T(t) = C_1\left(e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t} - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}(t_0-t)}\right) + C_3\frac{\cos\left(\frac{2t-t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}.$$

Taking into account the initial conditions (12), we find

$$\begin{cases} C_1\left(1 - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) + C_3\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)} = 0, \\ -C_1\left(1 - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right) + C_3\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)} = 0. \end{cases}$$

The received system determinant

$$\Delta = 2\left(1 - e^{\sqrt{\frac{\gamma-\lambda}{\beta}}t_0}\right)\cdot\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}$$

vanishes if $\lambda = \gamma$, or if $\lambda_n = \frac{\beta(\pi+2\pi n)^2 - \gamma t_0^2}{t_0^2}$, $n \in N$. Hence, $C_1 = C_3 = 0$ and $T(t) \equiv 0$, as these values for λ contradict the investigated case if $\forall t_0 \in \left(0, \frac{\pi\sqrt{\beta}}{\sqrt{2\gamma}}\right)$.

Case 4: $\lambda = \gamma$. If the indicated value is λ, (16) becomes as follows

$$T(t) = C_1 t + C_2 + C_3\cos\left(t\sqrt{\frac{2\gamma}{\beta}}\right) + C_4\sin\left(t\sqrt{\frac{2\gamma}{\beta}}\right).$$

Applying the same reasoning as in case 3, it is easy to verify that, again, the equation (11) and homogenous initial conditions are true if and only if $T(t) \equiv 0$.

Case 5. If $\lambda > \gamma$ (16) becomes as follows:

$$T(t) = C_1\cos\left(t\sqrt{\frac{\lambda-\gamma}{\beta}}\right) + C_2\sin\left(t\sqrt{\frac{\lambda-\gamma}{\beta}}\right) + C_3\cos\left(t\sqrt{\frac{\gamma+\lambda}{\beta}}\right) + C_4\sin\left(t\sqrt{\frac{\gamma+\lambda}{\beta}}\right).$$

From here, taking into consideration (11), we find

$$\cos\left(\frac{2t-t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)\left[C_1\cos\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)+C_2\sin\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)\right]+$$

$$+\sin\left(\frac{2t-t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)\left[C_3\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)-C_4\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)\right]=0.$$

Hence, $C_1=-C_2tg\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)$, $C_3=C_4ctg\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)$. Then

$$T(t)=C_1\frac{\sin\left(\frac{2t-t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)}+C_3\frac{\cos\left(\frac{2t-t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}.$$

Satisfying the obtained expression for $T(t)$ boundary conditions (12), we obtain:

$$\begin{cases} -C_1\frac{\sin\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)}+C_3\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}=0, \\ C_1\frac{\sin\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)}+C_3\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}=0. \end{cases}$$

From here we obtain:

$$\Delta=-2\frac{\sin\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{\lambda-\gamma}{\beta}}\right)\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma+\lambda}{\beta}}\right)}.$$

Hence, $\Delta=0$ if $\lambda_{1n}=\frac{(\pi+2\pi n)^2\beta-\gamma t_0^2}{t_0^2}$, or if $\lambda_{2n}=\frac{4\pi^2n^2\beta+\gamma t_0^2}{t_0^2}$, $n\in\mathrm{N}$. Hence, the problem (11), (12) has eigenvalues $\lambda_{1n}=\frac{(\pi+2\pi n)^2\beta-\gamma t_0^2}{t_0^2}$, $\lambda_{2n}=\frac{4\pi^2n^2\beta+\gamma t_0^2}{t_0^2}$ and eigenfunctions $T_{1n}(t)=C_{1n}\cos\left(\frac{\pi+2\pi n}{2t_0}t\right)$, $T_{2n}(t)=C_{2n}\sin\left(\frac{\pi n}{t_0}t\right)$, where C_{1n}, C_{2n} – correspond to them as arbitrary constants, which need definition.

Substituting the obtained eigenvalues $\lambda_{1n}=\frac{(\pi+2\pi n)^2\beta-\gamma t_0^2}{t_0^2}$ in equality

$$\alpha X''(x)-\lambda X(x)=0, \tag{17}$$

which is a derived relation (10), we reach the relation

$$\alpha X_n''(x)-\frac{(\pi+2\pi n)^2\beta-\gamma t_0^2}{t_0^2}X_n(x)=0.$$

The general solution to the latter is as follows:

$$X_n(x) = \alpha_{1n} \cos\left(x\sqrt{\frac{\gamma t_0^2 - (\pi + 2\pi n)^2 \beta}{\alpha t_0^2}}\right) + \beta_{1n} \sin\left(x\sqrt{\frac{\gamma t_0^2 - (\pi + 2\pi n)^2 \beta}{\alpha t_0^2}}\right),$$

where $\alpha_{1n}, \beta_{1n} = const$.

Then, solution $u_1(x,t)$ of problem (3)-(5), may be expressed as follows:

$$\overline{u}_1(x,t) = \sum_{n=1}^{\infty} \left[A_{1n} \cos\left(x\sqrt{\frac{\gamma t_0^2 - (\pi + 2\pi n)^2 \beta}{\alpha t_0^2}}\right) + B_{1n} \sin\left(x\sqrt{\frac{\gamma t_0^2 - (\pi + 2\pi n)^2 \beta}{\alpha t_0^2}}\right)\right] \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right), \tag{18}$$

where $\overline{u}_1(x,t)$ – corresponds to the eigenvalues λ_{1n}.

Conditions (5) make it possible to define A_{1n}, B_{1n}. Taking into consideration conditions 1), 2) of theorem 1, functions $\varphi_1(t)$ and $\varphi_2(t)$, $0 \le t \le t_0$ transform in Fourier series in eigenfuctions, i.e.:

$$\varphi_1(t) = \frac{\delta_0}{2} + \sum_{n=1}^{\infty} \delta_n \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right), \varphi_2(t) = \frac{\rho_0}{2} + \sum_{n=1}^{\infty} \rho_n \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right),$$

where

$$\delta_0 = \frac{2}{t_0} \int_0^{t_0} \varphi_1(t)\, dt, \rho_0 = \frac{2}{t_0} \int_0^{t_0} \varphi_2(t)\, dt,$$

$$\delta_n = \frac{2}{t_0} \int_0^{t_0} \varphi_1(t) \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right) dt, \rho_n = \frac{2}{t_0} \int_0^{t_0} \varphi_2(t) \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right) dt,$$

with converging $\sum\limits_{n=1}^{\infty} |\delta_n|$ and $\sum\limits_{n=1}^{\infty} |\rho_n|$ series.

Then, taking into consideration conditions (5), we obtain

$$\sum_{n=1}^{\infty} A_{1n} \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right) = \frac{\delta_0}{2} + \sum_{n=1}^{\infty} \delta_n \cos\left(\frac{\pi + 2\pi n}{2t_0} t\right),$$

$$\sum_{n=1}^{\infty}\left[A_{1n}\cos\left(x_0\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)+\right.$$

$$\left.+\ B_{1n}\sin\left(x_0\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)\right]\cos\left(\frac{\pi+2\pi n}{2t_0}t\right)$$

$$=\frac{\rho_0}{2}+\sum_{n=1}^{\infty}\rho_n\cos\left(\frac{\pi+2\pi n}{2t_0}t\right).$$

From here we find:

$$A_{1n}=\delta_n,$$

$$B_{1n}=\frac{\rho_n-\delta_n\cos\left(x_0\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)}{\sin\left(x_0\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)}.$$

Substituting values $A_{1n},\ B_{1n}$ in (18), we obtain:

$$\overline{u}_1(x,t)=\sum_{n=1}^{\infty}\frac{\delta_n\sin\left([x_0-x]\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)+\rho_n\sin\left(x\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)}{\sin\left(x_0\sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}\right)}\times$$

$$\times\cos\left(\frac{\pi+2\pi n}{2t_0}t\right). \tag{19}$$

We show that the function defined by equality (19) is the solution to the problem (3)-(5). Taking into consideration conditions 1), 2) of theorem 1, it is easily shown that Fourier coefficients δ_n and ρ_n do not exceed values $F_1\left(\frac{\pi+2\pi n}{2t_0}\right)^{-3}$ and $F_2\left(\frac{\pi+2\pi n}{2t_0}\right)^{-3}$ correspondingly, where $F_1, F_2 = const > 0$, $n = 1,\ 2,\ ....$

Indeed, using $\forall x \in (0, x_0)$ and $\forall t \in (0, t_0)$ it is easy to verify the truth of estimates:

$$|\overline{u}_1(x,t)|=\left|\frac{\delta_n\sin([x_0-x]\,\xi_n)+\rho_n\sin(x\xi_n)}{\sin(x_0\xi_n)}\cos\left(\frac{\pi+2\pi n}{2t_0}t\right)\right|\leq$$

$$\leq\left|\frac{\delta_n\sin([x_0-x]\,\xi_n)+\rho_n\sin(x\xi_n)}{\sin(x_0\xi_n)}\right|\leq|\delta_n|+|\rho_n|,$$

$$|\overline{u}_{1x}(x,t)| = \left|\frac{-\delta_n\xi_n\cos([x_0-x]\,\xi_n)+\rho_n\xi_n\cos(x\xi_n)}{\sin(x_0\xi_n)}\cos\left(\frac{\pi+2\pi n}{2t_0}t\right)\right| \le$$

$$\le \left|\frac{\delta_n\xi_n\cos([x_0-x]\,\xi_n)}{\sin(x_0\xi_n)}\right| + \left|\frac{\rho_n\xi_n\cos(x\xi_n)}{\sin(x_0\xi_n)}\right| \le (F_1+F_2)\left(\frac{\pi+2\pi n}{2t_0}\right)^{-2},$$

$$|\overline{u}_{1t}(x,t)| = \left|\frac{\delta_n\sin([x_0-x]\,\xi_n)+\rho_n\sin(x\xi_n)}{\sin(x_0\xi_n)}\frac{\pi+2\pi n}{2t_0}\sin\left(\frac{\pi+2\pi n}{2t_0}t\right)\right| \le$$

$$\le \left|\frac{\delta_n\frac{\pi+2\pi n}{2t_0}\sin([x_0-x]\,\xi_n)}{\sin(x_0\xi_n)}\right| + \left|\frac{\rho_n\frac{\pi+2\pi n}{2t_0}\sin(x\xi_n)}{\sin(x_0\xi_n)}\right| \le (F_1+F_2)\left(\frac{\pi+2\pi n}{2t_0}\right)^{-2},$$

$$|\overline{u}_{1xx}(x,t)| = \left|\frac{\delta_n\xi_n^2\sin([x_0-x]\,\xi_n)+\rho_n\xi_n^2\sin(x\xi_n)}{\sin(x_0\xi_n)}\cos\left(\frac{\pi+2\pi n}{2t_0}t\right)\right| \le$$

$$\le \left|\frac{\delta_n\xi_n^2\sin([x_0-x]\,\xi_n)}{\sin(x_0\xi_n)}\right| + \left|\frac{\rho_n\xi_n^2\sin(x\xi_n)}{\sin(x_0\xi_n)}\right| \le (F_1+F_2)\left(\frac{\pi+2\pi n}{2t_0}\right)^{-1},$$

$$|\overline{u}_{1xt}(x,t)|$$

$$= \left|\frac{-\delta_n\xi_n\cos([x_0-x]\,\xi_n)+\rho_n\xi_n\cos(x\xi_n)}{\sin(x_0\xi_n)}\frac{\pi+2\pi n}{2t_0}\sin\left(\frac{\pi+2\pi n}{2t_0}t\right)\right| \le$$

$$\left|\frac{\delta_n\frac{\pi+2\pi n}{2t_0}\xi_n\cos([x_0-x]\,\xi_n)}{\sin(x_0\xi_n)}\right| + \left|\frac{\rho_n\frac{\pi+2\pi n}{2t_0}\xi_n\cos(x\xi_n)}{\sin(x_0\xi_n)}\right| \le (F_1+F_2)\left(\frac{\pi+2\pi n}{2t_0}\right)^{-1},$$

$$|\overline{u}_{1tt}(x,t)| = \left|\frac{\delta_n\sin([x_0-x]\,\xi_n)+\rho_n\sin(x\xi_n)}{\sin(x_0\xi_n)}\left(\frac{\pi+2\pi n}{2t_0}\right)^2\cos\left(\frac{\pi+2\pi n}{2t_0}t\right)\right| \le$$

$$\left|\frac{\delta_n\left(\frac{\pi+2\pi n}{2t_0}\right)^2\sin([x_0-x]\,\xi_n)}{\sin(x_0\xi_n)}\right| + \left|\frac{\rho_n\left(\frac{\pi+2\pi n}{2t_0}\right)^2\sin(x\xi_n)}{\sin(x_0\xi_n)}\right|$$

$$\le (F_1+F_2)\left(\frac{\pi+2\pi n}{2t_0}\right)^{-1},$$

where $\xi_n = \sqrt{\frac{\gamma t_0^2-(\pi+2\pi n)^2\beta}{\alpha t_0^2}}$.

Thus, we conclude that function $\overline{u}_1(x,t)$ and its derivatives, appearing in the equation (1) are defined by converging series.

Next we investigate eigenvalues $\lambda_{2n} = \frac{4\pi^2 n^2 \beta + \gamma t_0^2}{t_0^2}$ within case 5. Substituting λ_{2n} in (17) we obtain the relation

$$\alpha X_n^{''}(x) - \frac{4\pi^2 n^2 \beta + \gamma t_0^2}{t_0^2} X_n(x) = 0.$$

The general solution to the latter may be expressed as follows:

$$X_n(x) = \alpha_{2n} \cos\left(x\sqrt{\frac{-4\pi^2 n^2 \beta - \gamma t_0^2}{\alpha t_0^2}}\right) + \beta_{2n} \sin\left(x\sqrt{\frac{-4\pi^2 n^2 \beta - \gamma t_0^2}{\alpha t_0^2}}\right),$$

where $\alpha_{2n}, \beta_{2n} = const$.

Then solution $u_1(x,t)$ of problem (3)-(5), may be expressed as follows:

$$\widetilde{u}_1(x,t) = \sum_{n=1}^{\infty} \left[A_{2n} \cos\left(x\sqrt{\frac{-4\pi^2 n^2 \beta - \gamma t_0^2}{\alpha t_0^2}}\right) + \right.$$

$$\left. + B_{2n} \sin\left(x\sqrt{\frac{-4\pi^2 n^2 \beta - \gamma t_0^2}{\alpha t_0^2}}\right)\right] \sin\left(\frac{\pi n}{t_0} t\right), \tag{20}$$

where $\widetilde{u}_1(x,t)$ – correspond to eigenvalues λ_{2n}.

Taking into consideration conditions 1), 2) of theorem 1, functions $\varphi_1(t)$ and $\varphi_2(t)$, $0 \le t \le t_0$ transform in Fourier series in eigenfunctions, i.e.:

$$\varphi_1(t) = \sum_{n=1}^{\infty} \theta_n \sin\left(\frac{\pi n}{t_0} t\right), \varphi_2(t) = \sum_{n=1}^{\infty} \vartheta_n \sin\left(\frac{\pi n}{t_0} t\right),$$

where

$$\theta_n = \frac{2}{t_0} \int_0^{t_0} \varphi_1(t) \sin\left(\frac{\pi n}{t_0} t\right) dt, \vartheta_n = \frac{2}{t_0} \int_0^{t_0} \varphi_2(t) \sin\left(\frac{\pi n}{t_0} t\right) dt,$$

with converging $\sum\limits_{n=1}^{\infty} |\theta_n|$ and $\sum\limits_{n=1}^{\infty} |\vartheta_n|$ – series.

Applying (5), we obtain

$$\sum_{n=1}^{\infty} A_{2n} \sin\left(\frac{\pi n}{t_0} t\right) = \sum_{n=1}^{\infty} \theta_n \sin\left(\frac{\pi n}{t_0} t\right),$$

$$\sum_{n=1}^{\infty}\left[A_{2n}\cos\left(x_0\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)+B_{2n}\sin\left(x_0\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)\right]\times\sin\left(\frac{\pi n}{t_0}t\right)=\sum_{n=1}^{\infty}\vartheta_n\sin\left(\frac{\pi n}{t_0}t\right).$$

From here we find:

$$A_{2n}=\theta_n,$$

$$B_{2n}=\frac{\vartheta_n-\theta_n\cos\left(x_0\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)}{\sin\left(x_0\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)}.$$

Substituting values A_{2n}, B_{2n} in (20), we obtain the result

$$\widetilde{u}_1=\sum_{n=1}^{\infty}\frac{\theta_n\sin\left([x_0-x]\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)+\vartheta_n\sin\left(x\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)}{\sin\left(x_0\sqrt{\frac{-4\pi^2n^2\beta-\gamma t_0^2}{\alpha t_0^2}}\right)}\sin\left(\frac{\pi n}{t_0}t\right).\tag{21}$$

Proof of convergence of series (21) and derivatives of $\widetilde{u}_{1x}(x,t)$, $\widetilde{u}_{1t}(x,t)$, $\widetilde{u}_{1xx}(x,t)$, $\widetilde{u}_{1xt}(x,t)$, $\widetilde{u}_{1tt}(x,t)$, appearing in equation (1) are is made similarly as with eigenvalues λ_{1n}.

Next, we investigate the problem (6)-(8). For this purpose we investigate eigenvalue and eigenfunction problems for the equation (17) with homogenous conditions

$$X(0)=X(x_0)=0.\tag{22}$$

It is plain that the general solution to equation (17) may be expressed as follows:

$$X(x)=\alpha_3e^{\sqrt{\frac{\lambda}{\alpha}}x}+\beta_3e^{-\sqrt{\frac{\lambda}{\alpha}}x}.\tag{23}$$

Here function $X(x)\neq 0$ only if $\lambda>0$. Indeed, if $\lambda>0$ (23) becomes:

$$X(x)=\alpha_3\cos\left(x\sqrt{-\frac{\lambda}{\alpha}}\right)+\beta_3\sin\left(x\sqrt{-\frac{\lambda}{\alpha}}\right).$$

Making the obtained expression for $X(x)$ satisfy boundary conditions (22), we obtain:

$$\begin{cases} X(0) = \alpha_3 = 0, \\ X(x_0) = \alpha_3 \cos\left(x_0\sqrt{-\frac{\lambda}{\alpha}}\right) + \beta_3 \sin\left(x_0\sqrt{-\frac{\lambda}{\alpha}}\right) = 0. \end{cases}$$

Hence, a non-trivial solution to the problem exists if

$$\lambda_{3n} = -\frac{\pi^2 n^2 \alpha}{x_0^2}.$$

These eigenvalues have the following corresponding eigenfunctions

$$X_n(x) = \alpha_{3n} \sin\left(\frac{\pi n}{x_0}x\right).$$

Substituting eigenvalues in (11) we obtain the relation

$$\beta T''(t) + \gamma T(t_0 - t) - \frac{\pi^2 n^2 \alpha}{x_0^2} T(t) = 0.$$

The general solution of the latter may be expressed as follows:

$$T(t) = C_1 \frac{\sin\left(\frac{t_0-2t}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)} + C_3 \frac{\cos\left(\frac{2t-t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)},$$

then

$$u_2(x,t) = \sum_{n=1}^{\infty}\left[A_{3n}\frac{\sin\left(\frac{2t-t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)} + B_{3n}\frac{\cos\left(\frac{2t-t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}\right] \times \sin\frac{\pi n}{x_0}x. \quad (24)$$

Functions $\varphi_3(x)$ and $\varphi_4(x)$, $0 \le x \le x_0$ may be defined in the segment $[-x_0, 0]$ via odd order. In this case each function transforms in Fourier series in eigenfunctions, i.e.:

$$\varphi_3(x) = \sum_{n=1}^{\infty} \omega_n \sin\left(\frac{\pi n}{x_0}x\right), \varphi_4(x) = \sum_{n=1}^{\infty} \upsilon_n \sin\left(\frac{\pi n}{x_0}x\right),$$

where

$$\omega_n = \frac{2}{x_0}\int\limits_0^{x_0} \varphi_3(x)\sin\left(\frac{\pi n}{x_0}x\right)dx, \upsilon_n = \frac{2}{x_0}\int\limits_0^{x_0} \varphi_4(x)\sin\left(\frac{\pi n}{x_0}x\right)dx, \qquad (25)$$

with converging $\sum\limits_{n=1}^{\infty}|\omega_n|$ and $\sum\limits_{n=1}^{\infty}|\upsilon_n|$ series.

Satisfying (24) condition (8), we obtain:

$$\sum_{n=1}^{\infty}\left[-A_{3n}\frac{\sin\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)} + B_{3n}\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}\right]\sin\left(\frac{\pi n}{x_0}x\right) =$$

$$= \sum_{n=1}^{\infty}\omega_n\sin\left(\frac{\pi n}{x_0}x\right).$$

$$\sum_{n=1}^{\infty}\left[A_{3n}\frac{\sin\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)}{\cos\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)} + B_{3n}\frac{\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}{\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}\right]\sin\left(\frac{\pi n}{x_0}x\right) =$$

$$= \sum_{n=1}^{\infty}\upsilon_n\sin\left(\frac{\pi n}{x_0}x\right).$$

From here we find:

$$A_{3n} = \frac{(\upsilon_n-\omega_n)\cos\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta x_0^2}}\right)}{2\sin\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta x_0^2}}\right)},$$

$$B_{3n} = \frac{(\omega_n+\upsilon_n)\sin\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}{2\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}.$$

Substituting values A_{3n}, B_{3n} in (24), we may write in result:

$$u_2(x,t) = \sum_{n=1}^{\infty}\left[\frac{(\upsilon_n-\omega_n)\sin\left(\frac{2t-t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)}{2\sin\left(\frac{t_0}{2}\sqrt{\frac{-\pi^2n^2\alpha-\gamma x_0^2}{\beta\, x_0^2}}\right)}+\right.$$

$$+\frac{(\omega_n+\upsilon_n)\cos\left(\frac{2t-t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}{2\cos\left(\frac{t_0}{2}\sqrt{\frac{\gamma x_0^2-\pi^2n^2\alpha}{\beta x_0^2}}\right)}\Bigg]\sin\left(\frac{\pi n}{x_0}x\right). \tag{26}$$

Proof that (26) is a solution to the problem (6)-(8), i.e. $u_2(x,t)$, $u_{2x}(x,t)$, $u_{2t}(x,t)$, $u_{2xx}(x,t)$, $u_{2xt}(x,t)$, $u_{2tt}(x,t)$ series are convergent.

By reason of conditions 1), 2) of theorem 1, integrating formula (25) by parts we obtain that Fourier coefficients ω_n and υ_n do not exceed values $F_5\left(\frac{\pi n}{x_0}\right)^{-3}$ and $F_6\left(\frac{\pi n}{x_0}\right)^{-3}$ correspondingly, where $F_5, F_6 = const > 0$, $n = 1,\ 2,\ ...,$ (26) series is majorized by convergent series.

Indeed, $\forall x \in (0, x_0)$ and $t \in (0, t_0)$ are true inequations

$$\begin{aligned}
|u_2(x,t)| &= \left|\left[\frac{(\upsilon_n-\omega_n)\sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)}+\frac{(\omega_n+\upsilon_n)\cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)}\right]\sin\left(\frac{\pi n}{x_0}x\right)\right| \\
&\le \left|\frac{(\upsilon_n-\omega_n)\sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)}+\frac{(\omega_n+\upsilon_n)\cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)}\right| \\
&\le \left|\frac{(\upsilon_n-\omega_n)\sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)}\right|+\left|\frac{(\omega_n+\upsilon_n)\cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)}\right| \\
&\le |\omega_n|+|\upsilon_n|,
\end{aligned}$$

$$\begin{aligned}
|u_{2x}(x,t)| &= \left|\left[\frac{(\upsilon_n-\omega_n)\sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)}+\frac{(\omega_n+\upsilon_n)\cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)}\right]\right. \\
&\quad \left.\frac{\pi n}{x_0}\cos\left(\frac{\pi n}{x_0}x\right)\right| \\
&\le \left|\frac{(\upsilon_n-\omega_n)\frac{\pi n}{x_0}\sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)}\right|+\left|\frac{(\omega_n+\upsilon_n)\frac{\pi n}{x_0}\cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)}\right| \\
&\le (|\omega_n|+|\upsilon_n|)\frac{\pi n}{x_0} \le (F_5+F_6)\left(\frac{\pi n}{x_0}\right)^{-2},
\end{aligned}$$

$$\begin{aligned}
|u_{2t}(x,t)| &= \left|\left[\frac{(\upsilon_n-\omega_n)\bar{\xi}_n\cos\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)}-\frac{(\omega_n+\upsilon_n)\bar{\eta}_n\sin\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)}\right]\right. \\
&\quad \left.\sin\left(\frac{\pi n}{x_0}x\right)\right|
\end{aligned}$$

$$\leq \left| \frac{(\upsilon_n - \omega_n)\,\bar{\xi}_n \cos\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} \right| + \left| \frac{(\omega_n + \upsilon_n)\,\bar{\eta}_n \sin\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right|$$

$$\leq (F_5 + F_6)\left(\frac{\pi n}{x_0}\right)^{-2},$$

$$|u_{2xx}(x,t)| = \left| \left[\frac{(\upsilon_n - \omega_n)\sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} + \frac{(\omega_n + \upsilon_n)\cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right] \left(\frac{\pi n}{x_0}\right)^2 \sin\left(\frac{\pi n}{x_0}x\right) \right|$$

$$\leq \left| \frac{(\upsilon_n - \omega_n)\left(\frac{\pi n}{x_0}\right)^2 \sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} \right| + \left| \frac{(\omega_n + \upsilon_n)\left(\frac{\pi n}{x_0}\right)^2 \cos\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right|$$

$$\leq (F_5 + F_6)\left(\frac{\pi n}{x_0}\right)^{-1},$$

$$|u_{2xt}(x,t)| = \left| \left[\frac{(\upsilon_n - \omega_n)\,\bar{\xi}_n \cos\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} - \frac{(\omega_n + \upsilon_n)\,\bar{\eta}_n \sin\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right] \frac{\pi n}{x_0} \cos\left(\frac{\pi n}{x_0}x\right) \right|$$

$$\leq \left| \frac{(\upsilon_n - \omega_n)\,\frac{\pi n}{x_0}\bar{\xi}_n \cos\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} \right| + \left| \frac{(\omega_n + \upsilon_n)\,\frac{\pi n}{x_0}\bar{\eta}_n \sin\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right|$$

$$\leq (F_5 + F_6)\left(\frac{\pi n}{x_0}\right)^{-1},$$

$$|u_{2tt}(x,t)| = \left| \left[\frac{(\upsilon_n - \omega_n)\,\bar{\xi}_n^2 \sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} + \frac{(\omega_n + \upsilon_n)\,\bar{\eta}_n^2 \sin\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right] \sin\left(\frac{\pi n}{x_0}x\right) \right|$$

$$\leq \left| \frac{(\upsilon_n - \omega_n)\,\bar{\xi}_n^2 \sin\left(\frac{2t-t_0}{2}\bar{\xi}_n\right)}{2\sin\left(\frac{t_0}{2}\bar{\xi}_n\right)} \right| + \left| \frac{(\omega_n + \upsilon_n)\,\bar{\eta}_n^2 \sin\left(\frac{2t-t_0}{2}\bar{\eta}_n\right)}{2\cos\left(\frac{t_0}{2}\bar{\eta}_n\right)} \right|$$

$$\leq (F_5 + F_6)\left(\frac{\pi n}{x_0}\right)^{-1}.$$

where $\bar{\xi}_n = \sqrt{\frac{-\pi^2 n^2 \alpha - \gamma x_0^2}{\beta\, x_0^2}}$, $\bar{\eta}_n = \sqrt{\frac{\gamma x_0^2 - \pi^2 n^2 \alpha}{\beta x_0^2}}$.

Summing up corresponding equalities (19) and (26), (21) and (26) we obtain the solution to problem 1 in region Ω.

Lemma 1. *If problem 1 possesses the solution, then it is unique if and only if*

$$\sin\left(\sqrt{\frac{\gamma t_0^2 - (\pi + 2\pi n)^2\, \beta}{\alpha t_0^2}} x_0\right) \neq 0,$$

$$\sin\left(\sqrt{\frac{-4\pi^2 n^2 \beta - \gamma t_0^2}{\alpha t_0^2}} x_0\right) \neq 0,$$

$$tg\left(\sqrt{\frac{-\pi^2 n^2 \alpha - \gamma x_0^2}{\beta x_0^2}} \frac{t_0}{2}\right) \cdot ctg\left(\sqrt{\frac{\gamma x_0^2 - \pi^2 n^2 \alpha}{\beta x_0^2}} \frac{t_0}{2}\right) \neq 0.$$

Indeed, in case of a uniform problem, i.e. where $\varphi_1(t) = \varphi_2(t) = 0$, $\varphi_3(x) = \varphi_4(x) = 0$, on the basis of equalities (19), (21), (26) and due to system completeness [2]

$$\left\{\cos\left(\frac{\pi + 2\pi n}{2t_0} t\right)\right\}_{n=1}^{\infty}, \left\{\sin\left(\frac{\pi n}{x_0} x\right)\right\}_{n=1}^{\infty}, \left\{\sin\left(\frac{\pi n}{t_0} t\right)\right\}_{n=1}^{\infty},$$

possesses $u(x,t) \equiv 0$. Hence, uniform problem 1 possesses a trivial solution only. Hence, we see that Lemma 1 is true.

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