International Journal of Differential Equations and Applications
Volume 4 No. 1 2002, 25-30

INFINITE DIMENSIONALITY OF THE SOLUTION SPACE OF $y' = 0$ ON NON-ARCHIMEDEAN FIELDS

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Abstract: The solution of the simplest differential equation $y' = 0$ on non-Archimedean fields is investigated. Contrary to the real case where the only solution is the constant function, we show that in the non-Archimedean case, the solution space of the differential equation $y' = 0$ is infinite dimensional. This is a unique property of non-Archimedean fields and is connected to the existence of nontrivial order preserving field automorphisms on such fields, contrary to the real case where the only order preserving field automorphism is the identity map.

AMS Subject Classification: 26E30, 12J25, 11D88
Key Words: Non-Archimedean calculus, Levi-Civita fields, differential equations

1. Introduction

Let K be a totally ordered non-Archimedean field extension of $\mathbb{R}$. We introduce the following terminology.

Received: April 17, 2001

Definition 1.1 ($\sim$, $\approx$, $\ll$, H, λ)**.** For $x, y \in K$, we say $x \sim y$ if and only if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$; for nonnegative $x, y \in K$, we say that x is infinitely smaller than y and write $x \ll y$ if and only if $nx < y$ for all $n \in \mathbb{N}$; finally, we say that x is approximately equal to y and write $x \approx y$ if and only if $x \sim y$ and $|x-y| \ll |x|$. We also set $\lambda(x) = [x]$, the class of x under the equivalence relation $\sim$.

The set of equivalence classes H (under the relation $\sim$) is naturally endowed with an addition via $[x]+[y] = [x \cdot y]$ and an order via $[x] < [y]$ if and only if $|y| \ll |x|$ (or $|x| \gg |y|$), both of which are readily checked to be well-defined. It follows that $(H, +, <)$ is a totally ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is the class [1]. The projection λ from K to H satisfies $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ and is a valuation.

The theorem of Hahn [4] provides a complete classification of any non-Archimedean extensions K of $\mathbb{R}$ in terms of their skeleton group H. In fact, invoking the axiom of choice it is shown that the elements of K can be written as formal power series over the group H with real coefficients, and the set of appearing "exponents" forms a well-ordered subset of H. The coefficient of the qth power in the Hahn representation of a given x will be denoted by $x[q]$, and the number d will be defined by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$. It is easy to check that $0 < d^q \ll 1$ if and only if $q > 0$ and $d^q \gg 1$ if and only if $q < 0$; moreover, $x \approx x[\lambda(x)]d^{\lambda(x)}$ for all $x \neq 0$.

From general properties of formal power series fields [8, 10], it follows that if H is divisible then K is real-closed. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [11], and for an overview of the related valuation theory the book by Krull [5]. A thorough and complete treatment of ordered structures can also be found in [9].

Throughout this paper, by K we will denote any totally ordered non-Archimedean field extension of $\mathbb{R}$. The smallest such field is the field of the formal Laurent series whose skeleton group is $\mathbb{Z}$; and the smallest such field that is both order complete and real-closed is the Levi-Civita field $\mathcal{R}$, first introduced by the Italian Mathematician Levi-Civita [6, 7]. In this case $H = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, the set of exponents in the Hahn representation of x is a left-finite subset of $\mathbb{Q}$, i.e. below any rational bound r there are only finitely many exponents. For a detailed study of the Levi-Civita field $\mathcal{R}$, we refer the reader to [1, 2, 3, 12, 13, 14, 15, 16].

2. Infinite Dimensionality of the Solution Space of $y' = 0$

In this section, we show that the differential equation $y' = 0$ has infinitely many linearly independent solutions on the interval $[-1, 1]$, and consequently the solution space is infinite dimensional.

Theorem 2.1. *The solution space of the differential equation $y' = 0$ on $[-1, 1]$ is infinite dimensional.*

Proof. For each $n \in \mathbb{N}$, let $g_n : [-1, 1] \to K$ be given by $g_n(x)[q] = x[q/(n + 1)]$. Then we show that, for all $n \in \mathbb{N}$, g_n is differentiable on $[-1, 1]$ with $g_n'(x) = 0$ for all $x \in [-1, 1]$.

So let $n \in \mathbb{N}$ be given. We first observe that $g_n(x+y) = g_n(x)+g_n(y)$ for all $x, y \in [-1, 1]$. Now let $x \in [-1, 1]$ and $\epsilon > 0$ in K be given. Let $\delta = \min\{\epsilon^2, d\}$, and let $y \in [-1, 1]$ be such that $0 < |y - x| < \delta$. Then

$$\left|\frac{g_n(y) - g_n(x)}{y - x}\right| = \left|\frac{g_n(y - x)}{y - x}\right| \sim |y - x|^n \text{ since } g(y - x) \sim (y - x)^{n+1}.$$

Since $|y - x| < \min\{\epsilon^2, d\}$, we obtain that $|y - x|^n \ll \epsilon$ both when ϵ is finite and when ϵ is infinitely small. Hence

$$\left|\frac{g_n(y) - g_n(x)}{y - x}\right| < \epsilon \text{ for all } y \in [-1, 1] \text{ satisfying } 0 < |y - x| < \delta.$$

It follows that g_n is differentiable at x, with $g_n'(x) = 0$. This is true for all $x \in [-1, 1]$ and for all $n \in \mathbb{N}$.

Hence g_n is a solution of the differential equation $y' = 0$ on $[-1, 1]$ for all $n \in \mathbb{N}$; and so is the constant function $g_0 : [-1, 1] \to K$, given by $g_0(x) = 1$ for all $x \in [-1, 1]$. Next we show that the set $S = \{g_n : n \in \mathbb{N} \cup \{0\}\}$ is linearly independent on $[-1, 1]$. It suffices to show that any finite subset of the infinite set S is linearly independent on $[-1, 1]$. So let $j \in \mathbb{N}$ and let $n_1 < n_2 < \cdots < n_j$ in $\mathbb{N} \cup \{0\}$ be given. We show that $g_{n_1}, g_{n_2}, \ldots, g_{n_j}$ are linearly independent on $[-1, 1]$. So suppose that $c_1 g_{n_1} + c_2 g_{n_2} + \cdots + c_j g_{n_j} = 0$ for some $c_1, c_2, \ldots, c_j$ in K;

we show that $c_1 = c_2 = \cdots = c_j = 0$. Since $c_1 g_{n_1} + c_2 g_{n_2} + \cdots + c_j g_{n_j} = 0$, we obtain that $c_1 g_{n_1}(d) + c_2 g_{n_2}(d) + \cdots + c_j g_{n_j}(d) = 0$. Hence

$$c_1 d^{n_1} + c_2 d^{n_2} + \cdots + c_j d^{n_j} = 0;$$

from which we infer that $c_1 = c_2 = \cdots = c_j = 0$. □

Remark 2.2. For each $n \in \mathbb{N}$, it is easy to check that the mapping g_n in the proof of Theorem 2.1 is an order preserving field automorphism on K; this is a sepcial property of non-Archimedean structures since it is well-known that the only order preserving field automorphism on $\mathbb{R}$ is the identity map [12].

Remark 2.3. The functions $g_n : [-1, 1] \to K$, for $n \in \mathbb{N} \cup \{0\}$, are not the only linearly independent solutions of $y' = 0$ on $[-1, 1]$. In fact, if $f : [-1, 1] \to K$ is any differentiable function on $[-1, 1]$, and if $n \in \mathbb{N} \cup \{0\}$, then we show that $f \circ g_n$ is also a solution of $y' = 0$. So let $x \in [-1, 1]$ be given. Then $(f \circ g_n)'(x) = f'(g_n(x))\, g_n'(x) = 0$. This is true for all $x \in [-1, 1]$; and hence $f \circ g_n$ is also a solution of $y' = 0$ on $[-1, 1]$.

Remark 2.4. The fact that there are infinitely many anti-derivatives for 0 makes integration much harder than in the real case; and this was one of the reasons people were discouraged from doing calculus or analysis on non-Archimedean fields. Most of the difficulties arise from the fact that such structures are totally disconnected in the order topology, and in recent years serious attempts were made to circumvent most of these difficulties and develop a calculus on these fields [1, 2, 3, 14, 15, 16]. In those papers, one of the following approaches was used, either to use a stronger concept of differentiability than the conventional one as in [1, 2, 3], or impose additional conditions on the function as in [16] or focus on classes of functions as in [14, 15].

Current research aims at developing an integration theory on non-Archimedean fields. We are investigating the merits of all three approaches mentioned above which we used to solve other problems like having an intermediate value theorem, an inverse function theorem, mean value theorem and analogues to other useful theorems of real calculus. We are trying to introduce a differentiability concept that would be strong enough to assure that the constant function is the only solution of the differential equation $y' = 0$. As an alternative solution, we may limit ourselves to a large class of functions that contain the continuations of all real useful functions and in which there is only one solution to $y' = 0$ under conventional differentiation.

3. Acknowledgments

For introducing me to the exciting field of non-Archimedean Analysis and for encouraging me to do my PhD dissertation on that topic, I would like to thank my former PhD advisor Professor Martin Berz. Also, for fruitful discussions and comments about non-Archimedean fields in general, I would like to thank Professor Sibylla Priess-Crampe and Professor Vera Zeidan.

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